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STATICALLY TAME PERIODIC HOMEOMORPHISMS OF COMPACT CONNECTED 3-MANIFOLDS. I. HOMEOMORPHISMS CONJUGATE TO ROTATIONS OF THE 3-SPHERE[1]

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ABSTRACT. Let f be a homeomorphism of the 3-sphere onto itself, of finite period n, and preserving orientation. Suppose that the fixed-point set F of f is a tame 1-sphere. It is shown that (1) the 3-sphere has a triangulation $K(\mathbf{S}^3)$ such that F forms a subcomplex of $K(\mathbf{S}^3)$ and f is simplicial relative to $K(\mathbf{S}^3)$. Suppose also that F is unknotted. It then follows that (2) f is conjugate to a rotation.

1. Statement of results. The following question has been proposed by R. H. Bing [$\mathbf{B}_1$]. Let f: $\mathbf{R}^3 \leftrightarrow \mathbf{R}^3$ be a periodic homeomorphism of Cartesian 3-space onto itself, with a straight line as its fixed-point set. Does it follow that f is conjugate to a rotation? We shall show that this conclusion does follow, if we assume also that f preserves orientation, or that the period of f is a prime. See Theorems 1.5 and 1.6 below. In the sequel to the present paper, we shall show that Bing's conjecture is true in general.

By an *n-manifold* we mean a separable metric space M in which each point has a neighborhood homeomorphic to Cartesian n-space $\mathbf{R}^n$. Thus M is not necessarily compact or connected. If M is separable and metric, and each point has a neighborhood which is an n-cell, then M is an *n-manifold with boundary*. Int M is the set of all points of M that have open Cartesian neighborhoods, and Bd M is the set of all points of M that do not.

Let M be a 3-manifold, and let A be a subset of M. Suppose that there is a triangulation K of M, and a homeomorphism h: $M \leftrightarrow M$, such that $h(A)$ is a polyhedron relative to K. Then A is *tame*. (For a general definition of a polyhedron, see the discussion just before Theorem 3.3 below.) By the Hauptvermutung [$\mathbf{M}_5$], this condition is independent of the choice of K, and so the above definition of tameness agrees with the classical definition, due to

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Artin and Fox [FA], for the cases in which M is $\mathbf{R}^3$ or the 3-sphere $\mathbf{S}^3$.

THEOREM 1.1. *Let M be a connected 3-manifold which is a subspace of $\mathbf{S}^3$. Let $f: M \leftrightarrow M$ be a homeomorphism of period n, preserving orientation and with fixed-point set F. Suppose that F is a 1-sphere which is the boundary of a tame 2-cell $D \subset \mathbf{S}^3$, and that f has period exactly n at each point of $M - F$. Let* Pr: $M \twoheadrightarrow \Omega$ *be the projection of M onto the orbit space Ω of f. Then Ω is a 3-manifold, and* Pr F *is tame in Ω.*

This is Theorem 11.8 below.

Let f: $M \leftrightarrow M$ be a periodic homeomorphism of a 3-manifold M onto itself. If for each i, the fixed-point set F_i of f^i is tame, then f is *statically tame*. (Thus the f of Theorem 1.1 is statically tame, since for each $i > 0$ we have $F_i = F$.) If M has a triangulation $K(M)$ relative to which f is simplicial, then f is *tame*. In the sequel to the present paper, it will be shown that if M is compact and connected, and f is statically tame, then f is tame. Theorem 1.1 is, in effect, a special case of this proposition, as the following theorem shows.

THEOREM 1.2. *Under the conditions of Theorem* 1.1, *M has a triangulation $K(M)$ such that F forms a subcomplex of $K(M)$ and f is simplicial relative to $K(M)$.*

This is a consequence of Theorem 1.1. Let $\tilde{U} = M - F$, and let $\rho = \mathrm{Pr}|\tilde{U}$: $\tilde{U} \twoheadrightarrow U \subset \Omega$, so that ρ is an n-sheeted covering. We know that Pr F is tame in Ω. Let $K(\Omega)$ be a triangulation of Ω in which Pr F forms a subcomplex, such that if $\sigma \in K(\Omega)$, and $\sigma \cap \mathrm{Pr}\, F \neq \varnothing$, then $\sigma \cap \mathrm{Pr}\, F$ is a face of σ. Let $K(M)$ be the lifting of $K(\Omega)$ to M. (To be exact, the simplices of $K(M)$ are the sets $\mathrm{Pr}^{-1}\sigma$ ($\sigma \in K(\Omega)$, $\sigma \subset F$) and the closures of the components of the nonempty sets $\mathrm{Pr}^{-1}(\sigma - \mathrm{Pr}\, F)$ ($\sigma \in K(\Omega)$).) Then f is simplicial relative to $K(M)$.

THEOREM 1.3. *Let f be a periodic homeomorphism $\mathbf{S}^3 \leftrightarrow \mathbf{S}^3$, of period n, preserving orientation, and suppose that the fixed-point set F of f is a 1-sphere. Then f has period exactly $= n$ at each point of $\mathbf{S}^3 - F$.*

The proof is in §2.

A tame 1-sphere in a 3-manifold M is *unknotted* if it is the boundary of a tame 2-cell.

THEOREM 1.4. *Let $K(M)$ be a triangulation of a 3-manifold M, and let F be an unknotted 1-sphere which is polyhedral relative to $K(M)$. Then F is the boundary of a 2-cell D which is polyhedral relative to M.*

The proof is in §12.

It was shown in [M] that if f: $\mathbf{S}^3 \leftrightarrow \mathbf{S}^3$ is a homeomorphism of period n, preserving orientation, and simplicial relative to some triangulation of $\mathbf{S}^3$, with an unknotted polygon as its fixed-point set, then f is conjugate to a rotation. (For a simpler proof, see P. A. Smith [$\mathbf{S}_2$].) Thus the preceding three theorems fit together to give the following.

THEOREM 1.5. *Let f be a periodic homeomorphism $\mathbf{S}^3 \leftrightarrow \mathbf{S}^3$, preserving orientation, and suppose that the fixed-point set of f is a tame unknotted 1-sphere. Then f is conjugate to a rotation.*

Obviously this is a restatement of Bing's conjecture, for the case in which orientation is preserved.

THEOREM 1.6. *Let f be a homeomorphism $\mathbf{S}^3 \leftrightarrow \mathbf{S}^3$, of prime period p, and suppose that the fixed-point set of f is a tame unknotted 1-sphere. Then f is conjugate to a rotation.*

To reduce this to Theorem 1.5, we need to show that the f of Theorem 1.6 preserves orientation. If p is odd, this is clear: if f reversed orientation, then every odd power of f would reverse orientation, which is impossible, because f^p is the identity. We suppose, then, that $p = 2$. It has been shown by Glen E. Bredon that if h: $N \leftrightarrow N$ is an involution of a compact connected n-manifold, and H is a component of the fixed-point set of h, then $n - \dim H$ is even if and only if G preserves local orientation at some point of H. (See Borel [$\mathbf{B}_4$, p. 79] or Bredon [$\mathbf{B}_5$].) In the present case, $n - \dim H = 3 - 1$. Since f preserves local orientation at some point, f preserves orientation, as desired.

2. Proof of Theorem 1.3. A classical result of P. A. Smith ([$\mathbf{S}_1$, p. 708]) asserts that if h: $\mathbf{S}^3 \leftrightarrow \mathbf{S}^3$ preserves orientation, has finite period (either prime or composite), and has a fixed point, then the fixed-point set H of h is a 1-sphere. Given f and n as in Theorem 1.3, suppose that f has period $k < n$ at some point of $\mathbf{S}^3 - F$. Then $k > 1$, k divides n, and $n = km$, where $1 < m < n$. Let $g = f^k$, and let G be the fixed-point set of g. Then g has period m, and F is a proper subset of G. This is impossible, because F and G are 1-spheres.

3. PL approximations and the Loop Theorem. Here we state various known results, some of them in modified forms.

THEOREM 3.1. *Every 3-manifold is triangulable. So also is every 3-manifold with boundary.*

(See Theorem 2 of [$\mathbf{M}_5$] and Theorem 9.1 of [$\mathbf{M}_8$]. See also Bing [$\mathbf{B}_2$].)

For each complex K, $|K|$ denotes the associated polyhedron. We shall need to observe the distinction between these, because often we shall be dealing with different triangulations $K_1(S)$, $K_2(S)$ of the same space S, in cases where there is no PL relation between $K_1(S)$ and $K_2(S)$. If $|K|$ is a manifold (or a manifold with boundary), then K will be called a PL *manifold* (or a PL *manifold with boundary*).

Let U be a topological space, and let ϕ be a function $U \to \mathbf{R}$, where $\mathbf{R}$ is the set of all real numbers. (Here ϕ is not necessarily continuous.) Suppose that for each compact subset A of U there is an $\varepsilon_A > 0$ such that $\phi(P) \geqslant \varepsilon_A$ for each $P \in A$. Then ϕ is *strongly positive*. Evidently every positive mapping is strongly positive. In a metric space, $d(P, Q)$ denotes the distance between P and Q.

THEOREM 3.2. *Let K and K' be PL 3-manifolds, let U be an open set in $|K|$, let g be a homeomorphism $U \to |K'|$, and let ϕ be a strongly positive function $U \to \mathbf{R}$. Then there is a homeomorphism g': $U \to |K'|$ such that*

(1) *g' is PL on every finite polyhedron in U (relative to subdivisions of K and K'),*

(2) *For each point P of U, $d(g(P), g'(P)) < \phi(P)$, and*

(3) *$g'(U) = g(U)$.*

This is Theorem 36.1, p. 253 of [**MGT**]. In the early literature, starting with [$\mathbf{M_5}$], this theorem was freely used but never proved; it was not shown that g' could be chosen so that $g'(U) = g(U)$.

Let K be a complex, and let S be a subset of $|K|$. Suppose that S has a triangulation $K(S)$ such that every simplex of $K(S)$ is a linear subsimplex of some simplex of K. Then S is a *polyhedron*, relative to K. If S is compact, then each such $K(S)$ is finite, and S is a *finite polyhedron*, relative to K. If S is a polyhedron relative to K, and is closed, then S forms a subcomplex of a subdivision of K, but for polyhedra in general this conclusion does not follow.

THEOREM 3.3. *Let K be a PL 3-manifold, let U be an open subset of $|K|$, let S be a subset of U, and suppose that U has a triangulation $K(U)$ in which S forms a subcomplex $K(S)$. Let ϕ be a strongly positive function $U \to \mathbf{R}$, and let V be an open set such that $S \subset V \subset U$. Then there is a homeomorphism g: $|K| \leftrightarrow |K|$ such that* (1) *$g(S)$ is a polyhedron relative to K,* (2) *for each point P of U, $d(P, g(P)) < \phi(P)$, and* (3) *$g|(|K| - V)$ is the identity.*

This is a variation on Theorem 5 of [$\mathbf{M_5}$]. (Similarly for Theorem 3.4 below.) It is an easy consequence of Theorem 3.2. Define ϕ' as a strongly positive function $V \to \mathbf{R}$ such that (a) $\phi'(P) \leqslant \phi(P)$ for every P and (b) for each P, $\phi'(P)$ is less than the distance between P and $|K| - V$. We can now apply Theorem 3.2 to the inclusion i: $V \leftrightarrow V$, $P \mapsto P$, using $K(U)$ as K, K as K', and ϕ' as ϕ. By (b), the resulting PL approximation i' can be extended to $|K| - V$ as the identity.

In a PL 3-manifold K, a set S is *tame* (relative to K) if there is a homeomorphism $|K| \leftrightarrow |K|$, mapping S onto a polyhedron (not necessarily a finite polyhedron). If there is an open set U containing S, and a homeomorphism $U \to |K|$, mapping S onto a polyhedron, then S is *semi-locally tame*. In the latter case it follows that U has a triangulation $K(U)$ relative to which S is a polyhedron. Thus Theorem 3.3 implies the following.

THEOREM 3.4. *In a PL 3-manifold, every semi-locally tame set is tame. In fact, if S is semilocally tame in K, and V is an open set containing S, then there is a homeomorphism* $g\colon |K| \leftrightarrow |K|$ *such that* (1) $g(S)$ *is a polyhedron relative to K and* (2) $g|(|K| - V)$ *is the identity.*

Here it is not required that S be compact, or even closed.

THEOREM 3.5 (THE LOOP THEOREM, FIRST FORM; CH. PAPAKYRIAKOPOULOS). *Let K be an orientable PL 3-manifold with boundary, and let* $M = |K|$. *If there is a PL loop in* Bd M *which is contractible in M but not in* Bd M, *then there is a PL 2-cell* Δ *such that* (1) Bd $\Delta = \Delta \cap$ Bd M *and* (2) Bd Δ *is not contractible in* Bd M.

Here a loop is a closed path without a distinguished base-point, that is, a mapping of a 1-sphere into a space. For proofs, see Papakyriakopoulos **[P]** and Stallings **[St]**. See also **[MGT**, pp. 182–190], where Stallings' proof is presented. This proof does not use orientability. The resulting generalized form of the Loop Theorem has the following consequence.

THEOREM 3.6 (THE LOOP THEOREM, SECOND FORM). *Let K be a PL 3-manifold, let* $M = |K|$, *and let* M^2 *be a connected polyhedral 2-manifold which is a closed set in* M^3, *such that* M^2 *is 2-sided in* M^3. *Suppose that there is a PL loop in* M^2 *which is contractible in* M^3 *but not in* M^2. *Then there is a PL 2-cell* Δ *in* M^3 *such that* (1) Bd $\Delta = \Delta \cap M^2$ *and* (2) Bd Δ *is not contractible in* M^2.

Here M^2 is *2-sided* in M^3 if M^2 separates a connected open neighborhood of M^2. It is easy to show, geometrically, that under these conditions, if U is such a connected neighborhood of M^2, then $U - M^2$ has exactly two components. Theorem 3.5 is simply a combination of Theorems 26.4 and 26.5 of **[MGT]**. If M^2 is the boundary of a polyhedral 3-manifold with boundary in M^3, then M^2 is always 2-sided in M^3, regardless of any question of orientability **[MGT**, Theorem 26.1, p. 191].

Note that in Theorem 3.5, M^2 is not required to be compact.

4. Solid tori and toroidal shells. Of the results in this section, some are old, some must be folklore, and some may be new.

THEOREM 4.1. *Let* M^2 *be a polyhedral 2-manifold, in a PL 3-manifold K. Let D be a polyhedral 2-cell in* M^2, *and let h be a PL homeomorphism* $M^2 \leftrightarrow M^2$, *such that* $h|(M^2 - D)$ *is the identity. Let* C *be a polyhedral closed neighborhood of* Int D. *Then* h *can be extended so as to give a PL homeomorphism* h': $|K| \leftrightarrow |K|$ *such that* $h'|(|K| - C)$ *is the identity.*

See [**MGT**, Theorem 27.1, p. 197].

Hereafter in this section, S will be a polyhedral solid torus, with Bd $S = T$. Let J be a 1-sphere in Int S. If there is a homeomorphism

$$\phi\colon \Delta \times S^1 \leftrightarrow S$$

(where Δ is a 2-cell and S^1 is a 1-sphere), such that $\phi(P \times S^1) = J$ for some $P \in$ Int Δ, then J is a *spine* of S. The following is obvious.

THEOREM 4.2. *If* J *is a spine of* S, *then* J *carries a generator of the fundamental group* $\pi(S)$; *and if* J *is a polygon, then* J *carries a generator of the* 1-*dimensional homology group* $H_1(S)$ (*with coefficients in the group* $\mathbf{Z}$ *of integers*).

By a *polygon* we mean a polyhedral 1-sphere. If J is a polygon in T, and J carries a generator of $H_1(S)$, then J is *longitudinal in* S. If J bounds a 2-cell in S, but does not bound a 2-cell in T, then J is *latitudinal in* S.

THEOREM 4.3. *Let* $J_1, J_2, \ldots, J_n$ *be a collection of disjoint polygons in* T. *If* $\bigcup_i J_i$ *carries a generator of* $H_1(S)$, *then* (1) *some* J_i *is longitudinal in* S *and* (2) *every* J_i *either is longitudinal in* S *or bounds a* 2-*cell in* T.

See [**M₄**, Lemmas 2 and 3]. Or see [**MGT**, Theorem 28.8, p. 204].

THEOREM 4.4. *Let* A *be a polyhedral annulus, and let* J *be a polygon in* Int A. *Then* (1) *bounds a* 2-*cell in* Int A *or* (2) J *carries a generator of* $H_1(A)$, *and decomposes* A *into two annuli* A_1, A_2, *containing the respective components of* Bd A, *and intersecting in* J.

This is a corollary of Theorem 27.1, p. 197 of [**MGT**].

THEOREM 4.5. *Let* J_1 *and* J_2 *be disjoint polygons in* T. *If* J_1 *is latitudinal in* S, *then* (1) J_2 *is latitudinal in* S *or* (2) J_2 *bounds a* 2-*cell in* T.

INDICATION OF PROOF. Let D be a polyhedral 2-cell in S, such that $J_1 = \text{Bd } D = D \cap T$. Split S apart at D. This gives a 3-cell C^3, with Bd C^3 the union of an annulus A and two homeomorphic copies of D. If J_2 bounds a 2-cell in A, then (2) holds; if not, then (1) holds.

Let J_1 and J_2 be polygons in a polyhedral orientable 2-manifold M^2. If $J_1 \cap J_2$ consists of a finite number of "true crossing points" of J_1 and J_2, then J_1 and J_2 are *in general position* (in M^2). We assign an orientation to M^2 (at random). To each polygon J in M^2 we can assign an orientation (at random). This gives a 1-cycle $Z^1(J)$ with constant coefficient 1. If J_1 and J_2 are in general position, then we can distinguish positive and negative crossings of J_1

and J_2. This gives the total crossing number $c = \mathrm{Cr}(Z^1(J_1), Z^1(J_2))$. If we reverse the orientation of M^2, or J_1, or J_2, then c is replaced by $-c$. Therefore $|c|$ is independent of the choices of these three orientations. If $|c| = 1$, then we say that J_2 crosses J_1 *algebraically once*.

THEOREM 4.6. *Let J_1 and J_2 be polygons in T, in general position. If J_1 is longitudinal and J_2 is latitudinal, then J_1 crosses J_2 algebraically once (and vice versa).*

INDICATION OF PROOF. Let D be a polyhedral 2-cell in S, such that $J_2 = \mathrm{Bd}\, D = D \cap T$. As in the proof of Theorem 4.5, we split S apart at D, getting a polyhedral 3-cell C^3, in which T appears as an annulus $A \subset \mathrm{Bd}\, C^3$. The components of $J_1 \cap \mathrm{Bd}\, C^3$ appear as disjoint broken lines $B \subset A$. If one of these has both its end-points in the same component of $\mathrm{Bd}\, A$, then it can be moved across $\mathrm{Bd}\, A$ into $\mathrm{Int}\, A$ by a cellular PL homeomorphism $T \leftrightarrow T$; and this homeomorphism can be extended to S. Thus this operation reduces the number of points in $J_1 \cap J_2$, preserving the stated properties of J_1 and J_2. Therefore we may suppose that each such B has its end-points in different components of $\mathrm{Bd}\, A$. If there are m such broken lines, then every 1-cycle $Z^1(J_1)$ (with constant coefficient 1) is homologous on S to a cycle of the form $\pm mZ_0^1$, where Z_0^1 is a generator of $H_1(S)$. Since J_1 is longitudinal, it follows that $m = 1$. The theorem follows.

THEOREM 4.7. *Let J be a latitudinal polygon in S, and let $J_1, J_2, \ldots, J_m$ be disjoint longitudinal polygons in S, such that J_i and J are in general position in T for each i. Then there is a PL homeomorphism $g\colon S \leftrightarrow S$ such that each polygon $g(J_i)$ intersects J in exactly one point (which is a "true crossing point").*

INDICATION OF PROOF. When we split T apart at J, T becomes an annulus A, and each J_i appears as finite union of disjoint broken lines B. As in the proof of the preceding theorem, the number of such broken lines can be reduced to 1, for each i; and this can be done without increasing the total number of points in the intersections $J \cap J_i$. This can be done by a PL homeomorphism $S \leftrightarrow S$. The theorem follows.

THEOREM 4.8. *Let T be a PL torus, and let J be a polygon in T which does not bound a 2-cell in T. Then $T - J$ is homeomorphic to the interior of an annulus. Thus, if T is split apart at J, giving a 2-manifold with boundary, the result is an annulus.*

INDICATION OF PROOF. Theorem 28.7, p. 204 of **[MGT]** asserts that if B is a regular neighborhood of such a J in T, then $\mathrm{Cl}(T - B)$ is an annulus. Now we apply Theorem 4.4 to J and the annulus B.

THEOREM 4.9. *Let S be a PL solid torus in a PL 3-manifold $M \subset \mathbf{S}^3$, and let J be a polygon in $T = \text{Bd } S$, such that J is contractible in $M - \text{Int } S$ but does not bound a 2-cell in T. Then J is longitudinal in S.*

PROOF. By Theorem 3.2 we may suppose that the imbedding $M \to \mathbf{S}^3$ is PL. Let A be an annulus which forms a regular neighborhood of J in T, and let

$$N = (\mathbf{S}^3 - S) \cup \text{Int } A,$$

so that N is a 3-manifold with boundary, and $\text{Bd } N = \text{Int } A$. Now J is contractible in $(M - S) \cup \text{Int } A$, and $(M - S) \cup \text{Int } A \subset N$. By the Loop Theorem (first form) it follows that there is a PL 2-cell Δ in N, with $\Delta \cap \text{Bd } N = \text{Bd } \Delta$, such that $J' = \text{Bd } \Delta$ is not contractible in N. Using Theorems 4.1 and 4.4 we can show that there is a PL homeomorphism $N \leftrightarrow N$, $\text{Bd } N \leftrightarrow \text{Bd } N$, $J' \leftrightarrow J$. Thus we may suppose that $\text{Bd } \Delta = J$. Theorem 28.12, on p. 205 of [**MGT**], asserts that under these conditions, J is longitudinal. (Note that without the hypothesis $M \subset \mathbf{S}^3$, this theorem would be trivially false; for example, it fails in every 3-manifold that contains a polyhedral projective plane.)

Let T and T' be disjoint homeomorphic 2-manifolds, in a PL 3-manifold K. Suppose that there is a 3-manifold W with boundary, lying in $|K|$, such that $\text{Bd } W = T \cup T'$, and W is homeomorphic to $T \times [0, 1]$. Then T and T' are *parallel* (or *concentric*) (in $|K|$).

THEOREM 4.10. *In an orientable PL 3-manifold K, let F be a polygon. For $i = 1, 2$, let T_i be a torus (not necessarily polyhedral) such that T_i is the frontier of a closed neighborhood S_i of F. Suppose that*

(1) *S_1 lies in the interior of a regular neighborhood N of F (relative to a subdivision of K),*

(2) *$S_2 \subset S_1 - T_1$, and,*

(3) *For $i = 1, 2$ there is an open set V_i containing T_i, and a triangulation $K(V_i)$ of V_i, such that T_i forms a subcomplex of $K(V_i)$.*

Then

(4) *For $i = 1, 2$, S_i is a solid torus, with F as a spine and*

(5) *T_1 and T_2 are parallel.*

A set homeomorphic to $T \times [0, 1]$ (where T is a torus) will be called a *toroidal shell.*

PROOF. By two applications of Theorem 3.3, the theorem reduces to the case in which T_1 and T_2 are polyhedra (relative to K). Let S_3 and S_4 be regular neighborhoods of F, in subdivisions of K, such that $S_3 \subset \text{Int } S_2$ and $S_4 \subset \text{Int } S_3$, and let $T_i = \text{Bd } S_i$ $(i = 3, 4)$. It is elementary to observe that $\text{Bd } N$ and T_3 are parallel, and that T_1 separates $\text{Bd } N$ from T_3 in $N - \text{Int } S_3$. A special case of a theorem of C. H. Edwards ([E, p. 414, Theorem 9]) asserts that under these conditions T_1 is parallel to Bd N and to T_3. By repeated applications of Edwards' theorem it follows that there is a homeomorphism $N - \text{Int } S_4 \leftrightarrow \text{Bd } N \times [0, 1]$, mapping each set Bd N, T_i onto a slice Bd $N \times k$. Therefore, for $i = 1, 2$ there is a homeomorphism g_i: $N \leftrightarrow S_i$ such that $g_i | S_4$ is the identity. Thus (4) holds. Similarly, there is a homeomorphism h: $S_1 \leftrightarrow N$, $S_2 \leftrightarrow S_3$, so that (5) holds.

I am indebted to the referee for the reference to Edwards, and for the scheme of the above proof.

THEOREM 4.11. *In a PL 3-manifold M, let A be a polyhedral annulus, let J and J' be polygons in* Int A, *each of which separates the components of* Bd A *from one another in A, and let W be a neighborhood of* Int A *in M. Then there is a PL homeomorphism h: $M \leftrightarrow M$, $A \leftrightarrow A$, $J \leftrightarrow J'$, such that $h|(M - W)$ and $h|$*Bd A *are identity mappings.*

PROOF. Theorem 27.2, p. 197 of **[MGT]**, asserts that under these conditions we can move J onto J' by a PL homeomorphism h' such that $h'|$Bd A is the identity and h' is the composition of a finite sequence of cellular homeomorphisms. Now we use Theorem 4.1 repeatedly to get the desired h.

The following is needlessly special, but is sufficient for our present purposes.

THEOREM 4.12. *Let J be a polygon, in a polyhedral torus T, such that J does not bound a 2-cell in T. Then J is not contractible in T.*

PROOF. Let S be a polyhedral solid torus, with Bd $S = T'$. By Theorem 4.8 we know that when T is split apart at J, the result is an annulus A. Therefore there is a mapping h: $A \twoheadrightarrow T'$, such that h "reidentifies the components of Bd A," and maps these onto a longitudinal polygon J in T'. Thus there is a homeomorphism $T \leftrightarrow T'$, $J \leftrightarrow J'$. Since J' is not contractible in T', J is not contractible in T.

Note that the proof proves more than the theorem; it shows that a polygon can be imbedded in a PL torus in only two topologically different ways.

5. Invariant toroidal neighborhoods of F. Let $M \subset S^3, f, F$, and n be as in Theorem 1.1. By Theorem 1.3 we know that f has period exactly n at each point of $M - F$. Let Ω be the orbit space of f, with the usual topology, and let

$$\text{Pr}: M \twoheadrightarrow \Omega$$

be the projection. Let $K(M)$ be a triangulation of M in which F forms a subcomplex. Let

$$\tilde{U} = M - F, \quad U = \text{Pr } \tilde{U}, \quad \rho = \text{Pr}|\tilde{U}.$$

Then ρ: $\tilde{U} \twoheadrightarrow U$ is an n-sheeted covering. Since U is a 3-manifold, it follows that U has a triangulation $K(U)$. We may suppose that the diameters

$\delta\sigma$ of the simplices σ of $K(U)$ approach 0 as σ approaches Pr F, in the sense that for every $\varepsilon > 0$ there is a neighborhood N_ε of Pr F such that if σ intersects N_ε, then $\delta\sigma < \varepsilon$. (This property is obtainable by subdivision. Deductively, it will not be needed, but to omit it might seem unnatural.)

Let $K(\tilde{U})$ be the lifting of $K(U)$, that is, the set of all components of all sets of the form $\rho^{-1}(\sigma)$, where $\sigma \in K(U)$. Then $K(\tilde{U})$ is a triangulation of $\tilde{U}$, and $f|\tilde{U}$ is simplicial relative to $K(\tilde{U})$. Note that we have no reason to suppose that the simplices of $K(\tilde{U})$ are polyhedral relative to $K(M)$.

A subset A of M will be called *f-invariant* if $f(A) = A$.

THEOREM 5.1. *Every open neighborhood X of F contains a solid torus $\tilde{S}_X$ such that* (1) *$\tilde{S}_X$ is f-invariant,* (2) Bd *$\tilde{S}_X$ is polyhedral relative to $K(\tilde{U})$, and* (3) *F is a spine of $\tilde{S}_X$.*

For an indication of the way in which this theorem will be used, see the remarks just after the proof.

PROOF. Evidently there is no loss of generality in supposing that $\overline{X}$ is a regular neighborhood of F in a subdivision of $K(M)$. Let S be a regular neighborhood of F, in a subdivision of $K(M)$, chosen sufficiently small so that

$$\bigcup_i f^i(S) \subset X.$$

We assert that Pr F has arbitrarily small closed connected neighborhoods S_X such that $T = \text{Fr } S_X$ is a 2-manifold which is polyhedral relative to $K(U)$. (For example, take a small closed neighborhood W of Pr F, such that Fr W is a polyhedron, and then add to W a small regular neighborhood of Fr W. Note that we write Fr S_X, not Bd S_X: not for a long time will we know that S_X is a 3-manifold with boundary.) If T is not connected, then we can make it connected, simply by boring holes in $S_X - \text{Pr } F$, from one component of T to another. Thus we may suppose that

(a) $T = \text{Fr } S_X$ is a connected 2-manifold which is polyhedral relative to $K(U)$.

We choose S_X so that (b) $T \subset \text{Pr Int } S$ and (c) the set

$$T - \text{Pr} \bigcap_i f^i(\text{Int } S)$$

lies in a finite union of disjoint 2-cells D_j. (At the outset, we get (b) and (c) simply by requiring that (b′) $T \subset \text{Pr} \bigcap_i f^i(\text{Int } S)$. But (b′) would not necessarily be preserved by certain hypothetical operations presently to be described.) Evidently T is 2-sided in U, because $S_X - \text{Pr } F$ is a 3-manifold with boundary, and T is its frontier in U [**MGT**, Theorem 26.1, p. 191].

Suppose that Δ is a polyhedral 2-cell in Pr(Int $S - F$), such that Bd $\Delta = \Delta \cap T$ and Bd Δ is not contractible in T. Then Δ will be called a *Loop Theorem*

Disk (LTD) in the pair [Pr S, T]. If there is such an LTD Δ, then Δ can be chosen so as not to intersect any of the 2-cells D_j; any given Δ can be moved off the D_j's, preserving its stated properties. If $\Delta \cap D_j = \varnothing$ for each j, then Δ can be added to T, and the resulting 2-dimensional polyhedron can be split apart at Δ, giving a 2-manifold $T' = \text{Fr } S'_X$, where S'_X is a neighborhood of Pr F. There are now two cases to consider.

Case 1. T' is connected. Here S'_X is smaller or larger than S_X, according as $\Delta \subset S_X$ or Int $\Delta \cap S_X = \varnothing$. In either case, the 1-dimensional Betti number $p^1(T')$ (with integer coefficients) is $p^1(T) - 2$.

Case 2. T' is not connected, $= T_1 \cup T_2$. Since $\text{Pr}^{-1}(T')$ separates F from Bd S in S, some component C of $\text{Pr}^{-1}(T')$ has the same property. Let U be the component of $S - C$ that contains F, and let $S''_X = \text{Pr } \overline{U}$. Then Fr S''_X is T_1 or T_2, say, T_1; and since T_2 is not a 2-sphere, we have $p^1(T_1) < p^1(T) - 2$. We replace S_X by S''_X, and T by T_1.

Under the above conditions for Δ, the splitting operation preserves not only (a) and (b) but also (c). And iterations of the operation must terminate. Thus we may assume hereafter that we have S_X and T satisfying (a), (b), (c), and also

(d) The pair [Pr S, T] contains no LTD.

Let

$$\tilde{S}_X = \text{Pr}^{-1} S_X, \qquad \tilde{T} = \text{Pr}^{-1} T = \text{Bd } \tilde{S}_X.$$

LEMMA 5.1.1. $\tilde{S}_X \subset X$.

PROOF. Since $T \subset \text{Pr Int } S$, we have

$$\tilde{T} = \text{Pr}^{-1} T \subset \text{Pr}^{-1}\text{Pr Int } S = \bigcup_i f^i(\text{Int } S) \subset X.$$

Therefore $\tilde{S}_X \subset X$.

LEMMA 5.1.2. *$\tilde{T}$ is connected.*

PROOF. $\tilde{S}_X$ is connected, because every component of $\tilde{S}_X$ contains F. Since $\overline{X}$ is a regular neighborhood of a polygon, $M - X$ is connected. Let Y be the component of $M - \tilde{S}_X$ that contains $M - X$. Since $\tilde{S}_X$ is f-invariant, so also is $\overline{Y}$; $\overline{Y}$ is a PL 3-manifold with boundary; and Bd $\overline{Y} \subset \tilde{T}$. Since $\tilde{S}_X$ is connected, so also is Bd $\overline{Y}$; Bd $\overline{Y}$ is a component of $\tilde{T}$, and is f-invariant. Thus Pr Bd $\overline{Y}$ is a component of T. If B were another component of $\tilde{T}$, then Pr B would be another component of T, which is impossible, because T is connected. Therefore $\tilde{T}$ is connected, which was to be proved.

LEMMA 5.1.3. *Let i be the inclusion $\tilde{T} \to X - F$. Then the induced homomorphism $i^*\colon \pi(\tilde{T}) \to \pi(X - F)$ is injective.*

PROOF. Let $\tilde{p}$ be a closed path in $\tilde{T}$, and let $p = \Pr(\tilde{p})$. Let $\{D_j\}$ be as in condition (c) above. Then p can be moved, by a homotopy, to give a closed path p' in $T \cap \Pr \bigcap_i f^i(\text{Int } S)$, and the homotopy between p and p' in $\pi(T)$ can be lifted so as to give a closed path $\tilde{p}'$, in $\tilde{T} \cap \bigcap_i f^i(\text{Int } S)$, such that $\tilde{p}' \simeq \tilde{p}$ in $\pi(\tilde{T})$.

Thus, if the lemma is false, there is a closed path $\tilde{p}'$ in $\tilde{T} \cap \bigcap_i f^i(\text{Int } S)$ such that $\tilde{p}'$ is contractible in X but not in $\tilde{T}$. Since $\overline{X}$ and S are regular neighborhdoods of F in K, with $S \subset X$, it follows that S is a retract of $\overline{X}$. (Use Theorem 4.10, or some much simpler theorem.) Therefore $\tilde{p}'$ is contractible in $S - F$, and hence in Int $S - F$. Therefore $p' = \Pr(\tilde{p}')$ is contractible in Pr(Int $S - F$). But since $(\Pr|T)^*$: $\pi(\tilde{T}) \to \pi(T)$ is injective, p is not contractible in T. But since $T = \text{Fr } S_X$, it follows that T is 2-sided in the 3-manifold Pr(Int $S - F$). Therefore Theorem 3.5 applies, and gives us a contradiction of condition (d) for S and T.

LEMMA 5.1.4. *$\tilde{T}$ is a torus.*

PROOF. Since $\pi(X - F) \approx \pi(\text{Fr } X) \approx \mathbf{Z} + \mathbf{Z}$, it follows by the preceding lemma that $\pi(\tilde{T})$ is commutative. Therefore T is either a torus or a 2-sphere. Now $\tilde{T}$ is polyhedral relative to $K(\tilde{U})$. By Theorem 3.3 (as in the proof of Theorem 4.10), there is a homeomorphism h: $M \leftrightarrow M$ such that $h(\tilde{T})$ is a polyhedron relative to $K(M)$ and h is the identity except at points of $X - F$. Thus, if $\tilde{T}$ is a 2-sphere, then F lies in a 3-cell in X, which is impossible. The lemma follows.

By Theorem 4.10 it follows that $\tilde{S}_X$ is a solid torus. This completes the proof of Theorem 5.1.

Nearly all of the rest of this paper will be devoted to the proof of Theorem 11.5, which asserts, in effect, that the $\tilde{S}_X$ of Theorem 5.1 can be chosen so that Pr $\tilde{S}_X$ is a solid torus. (The S_1 of Theorem 11.5 is a suitably chosen $\tilde{S}_X$ as in Theorem 5.1.) In the proof of Theorem 11.5 we form a "standard" solid torus S_1', and decompose it into an infinite collection of solid tori, plus a spine F'. Thus the purpose of §§6–11 below is to define a neighborhood S_1 of F such that Pr S_1 has a similar decomposition; this will enable us to define the desired homeomorphism in the proof of Theorem 11.5. The proof of Theorem 1.1 will then become easy.

6. Eliminating cellular oscillations. We resume the proof of Theorem 1.1. In §§6–11, the definitions and notations of §5, up to and including Theorem 5.1, will be regarded as conventions, and used without reference or comment.

The next step in the proof of Theorem 1.1 is to show that every neighborhood of F contains a solid torus S, satisfying the conditions for $\tilde{S}_X$ in Theorem 5.1, such that S "approximates the shape of a regular neighborhood of F." A usable definition of the latter idea will emerge in this and later sections. Meanwhile it is intuitively evident that a solid-toroidal neighborhood of F may fail, in at least two ways, to have the desired property.

(a) If J is a polygon in the boundary of a regular neighborhood N of F, and J is of small diameter, and bounds a 2-cell in Bd N, then J bounds a 2-cell of small diameter in Bd N. If S fails to have this property, then we say that Bd S *has cellular oscillations*.

(b) If J and J' are disjoint latitudinal polygons in Bd N, and $J \cup J'$ is of small diameter, then $J \cup J'$ bounds an annulus of small diameter in Bd N. If S fails to have this property, then we say that Bd S *has annular oscillations*.

We shall show, in effect, that these are the only possible difficulties, and that they can be dealt with separately, in the stated order.

Let N be a regular neighborhood of F, relative to a subdivision $K'(M)$ of $K(M)$. Then N has a natural decomposition into dual cells D_v, which are neighborhoods of the points v of F which are vertices of $K'(M)$, such that

(1) Two sets D_v, D_w intersect only if v and w are the end-points of an edge of $K'(M)$ and

(2) Each nonempty intersection $D_v \cap D_w = E_{vw}$ is a PL 2-cell, lying in Bd $D_v \cap$ Bd D_w and intersecting F in exactly one point.

The 2-cells E_{vw} are "orthogonal to the edges vw." They are called *splitting disks*.

Let N_0 and N_1 be regular neighborhoods of F, defined relative to subdivisions $K'(M)$ and $K''(M)$ of $K(M)$. Let

$$\mathbf{P} = \{P_1, P_2, \ldots, P_m\}$$

be a finite set of vertices of $K'(M)$, lying in F, such that the points P_j appear in the stated cyclic order on F. (We use integers modulo m as subscripts, here and in similar situations hereafter.) Suppose that no two points of $\mathbf{P}$ are consecutive on F in $K'(M)$, so that the dual cells $D_j = D_{P_j}$ of N_0 that contain the points P_j are disjoint. For each j, let D_j' be the dual cell of N_1 that contains P_j, and suppose that $K''(M)$ is a subdivision of $K'(M)$, sufficiently fine so that $N_1 \subset \text{Int}\, N_0$, and

$$\bigcup_i f^i(D_j') \subset \bigcap_i f^i(\text{Int}\, D_j)$$

for each j. It is geometrically clear that for each j there is a polyhedral 2-cell $E_j \subset D_j$ such that (1) Bd $E_j = E_j \cap$ Bd D_j, (2) E_j intersects no splitting disk of N_0 or N_1, (3) $E_j \cap$ Bd D_j' is a polygon, and (4) $E_j \cap F = P_j$. Thus E_j separates D_j (and D_j') into two disjoint connected sets each of which contains a point of $F \cap$ Bd D_j (and of $F \cap$ Bd D_j'). For each j, let

$$B_j = E_j - \text{Int}\, D_j'.$$

Then B_j is an annulus; the components of Bd B_j are polygons lying in Bd $N_0 \cap$ Bd D_j and Bd $N_1 \cap$ Bd D_j', and intersecting no splitting disk of N_0

or N_1, and such that

$$\text{Int } B_j \subset \text{Int } D_j - D_j'.$$

So far, all the sets mentioned are polyhedra relative to $K(M)$, but are not necessarily polyhedra relative to $K(\tilde{U})$; many of them are not even subsets of $\tilde{U}$.

Let $K_1(\tilde{U})$ be a triangulation of $\tilde{U}$, such that every simplex of $K_1(\tilde{U})$ is a linear subsimplex of some simplex of $K''(M)$, and such that $\delta\sigma$ ($\sigma \in K_1(U)$) approaches 0 as $\sigma \to F$. Let ϕ: $U \to \mathbf{R}$ be a strongly positive function. By Theorem 3.3 it follows that there is a homeomorphism

$$g: \tilde{U} \leftrightarrow \tilde{U}$$

such that g is PL relative to $K_1(\tilde{U})$ and $K(\tilde{U})$, and such that g is a ϕ-approximation of the identity, in the sense that for each point P we have $d(P, g(P)) < \phi(P)$. If ϕ is chosen so that $\phi(P)$ is always less than the distance from P to F ($= M - \tilde{U}$), then g can be extended to give a homeomorphism

$$g: M \leftrightarrow M,$$

such that $g|F$ is the identity. For each j, let

$$C_j = g(D_j), \quad C_j' = g(D_j'), \quad A_j = g(B_j), \quad d_j = g(E_j), \quad S_0 = g(N_0).$$

We take ϕ sufficiently small so that g preserves the stated relation between the D_j's and the D_j''s; that is,

$$\bigcup_i f^i(C_j') \subset \bigcap_i f^i(C_j).$$

Let

$$\mathbf{C} = \{C_j\}, \quad \mathbf{C}' = \{C_j'\}, \quad \mathbf{A} = \{A_j\}.$$

Then the system

$$\mathbf{B} = [\mathbf{P}, \mathbf{C}, \mathbf{C}', \mathbf{A}, S_0]$$

is a *barrier system for F*. The *mesh* $\delta\mathbf{B}$ of $\mathbf{B}$ is the maximum of the diameters of the sets C_j and the diameters of the components of $S_0 - \bigcup_j C_j$. Evidently the preceding discussion has proved nearly all of the following:

THEOREM 6.1. *Let N be a regular neighborhood of F, and let ε be a positive number. Then there is a barrier system*

$$\mathbf{B} = [\mathbf{P}, \mathbf{C}, \mathbf{C}', \mathbf{A}, S_0]$$

for F such that (1) $S_0 \subset N$ *and* (2) $\delta\mathbf{B} < \varepsilon$. *And given any finite subset* $\mathbf{Q}$ *of F,* $\mathbf{B}$ *can be chosen so that* (3) $\mathbf{Q} \subset \mathbf{P}$.

(To get this, we choose $\mathbf{P}$ so that $\mathbf{Q} \subset \mathbf{P}$, and take the points of $\mathbf{P}$ sufficiently close together so that the components of $F - \mathbf{P}$ have diameter less than ε. Then form $N_0 \subset N$, using a very fine subdivision $K'(M)$ of $K(M)$, and take ϕ very close to 0.)

DEFINITION 6.2. Let $\mathbf{B} = [\mathbf{P}, \mathbf{C}, \mathbf{C}', \mathbf{A}, S_0]$ be a barrier system for F. Let S be a solid torus such that

(1) $F \subset \operatorname{Int} S$, $S \subset \operatorname{Int} S_0$,

(2) Bd S is polyhedral relative to $K(\tilde{U})$,

(3) $S \cap \bigcup_j A_j = \varnothing$, and

(4) Bd S is in general position relative to each set Bd C_j', in the sense that Bd $S \cap$ Bd C_j' is a finite union of disjoint polygons, at which Bd S and Bd C_j' pierce each other locally in $\tilde{U}$. Then $\mathbf{B}$ is a *barrier system for* S.

THEOREM 6.3. *Let* $\mathbf{B}$ *be a barrier system for* S. *Then each set* Bd $S \cap$ Bd C_j' *contains a polygon* J *which is latitudinal in* S.

PROOF. Bd $C_j' \cap$ Bd A_j separates Bd C_j' into two disjoint sets each of which is the interior of a 2-cell. Since $S \cap A_j = \varnothing$, it follows that the two points P, Q of $F \cap$ Bd C_j' lie in different components of $S \cap$ Bd C_j'.

Let M^2 be the union of (1) the component W of $S \cap$ Bd C_j' that contains P and (2) all components of $S \cap$ Bd C_j' that lie in components of Bd $C_j' - W$ that do not contain Q. Evidently W is a 2-cell with a finite number of holes (perhaps none). Let J be the "outer boundary" of W, that is, the component of Bd W that is the boundary of the closure of the component of Bd $C_j' - W$ that contains Q. We shall show that J is latitudinal in S.

LEMMA 6.3.1. *Every component of* Bd M^2, *other than* J, *bounds a* 2-*cell in* Bd S.

PROOF. J is the boundary of a 2-cell $D_J \subset$ Bd C_j', with $M^2 \subset D_J$. Suppose that the lemma is false, so that some component J' of Bd $M^2 - J$ does not bound a 2-cell in Bd S; and choose J' as the inmost polygon in D_J with the stated properties. It follows that J' is the boundary of a 2-cell d (PL relative to $K(\tilde{U})$) such that Int d lies in either Int $S - F$ or Int $S_0 - S$. The point is that J' is the outer boundary of a 2-cell with holes whose interior lies in Int $S - F$ or in Int $S_0 - S$; and since the boundaries of the holes bound 2-cells in Bd S, it follows that the holes can be filled, with 2-cells lying arbitrarily close to Bd S, so as to give the desired d.

By Theorem 4.10 we know that F is a spine of S and of S_0. Suppose that Int $d \subset$ Int $S - F$. Since J' does not bound a 2-cell in Bd S, it follows that J' is latitudinal in S, which is impossible, because $d \cap F = \varnothing$. Suppose that Int $d \subset$ Int $S_0 - S$. Then it follows by Theorem 4.9 that J' is longitudinal in S. Therefore J' carries a generator of $\pi(S)$, and the injection $\pi(S) \to \pi(S_0)$ annihilates $\pi(S)$. This is absurd, because F is a spine both of S and of S_0.

From the lemma it follows that J is the boundary of a 2-cell d, such that

Int $d \subset S$; the proof is the same as that of the lemma. (Every component of Bd $M - J$ bounds a 2-cell in Bd S, and so the holes in M can be filled with 2-cells lying in Int S.) Therefore J is latitudinal in S, which was to be proved.

THEOREM 6.4. *Let* $\mathbf{B} = [\mathbf{P}, \mathbf{C}, \mathbf{C}', \mathbf{A}, S_0]$ *be a barrier system for* S. *Then every polygon* J *which lies in a set* Bd $S \cap C_i$ *either is latitudinal in* S *or bounds a* 2-*cell in* Bd S.

PROOF. Take $j \neq i$, and let J_j be a polygon in Bd $S \cap$ Bd C_j' such that J_j is latitudinal in S. Let D_j be a 2-cell such that Bd $D_j = J_j$ and Int $D_j \subset$ Int S. When we "split S apart at D_j," we get a 3-cell C whose boundary is the union of two 2-cells D_j', D_j'' and an annulus A whose interior contains J. If J bounds homologically on A, then J bounds a 2-cell in A, and hence bounds a 2-cell in Bd S. If J separates D_j' from D_j'' in Bd C, then J is latitudinal in S.

THEOREM 6.5. *Let* $\mathbf{B} = [\mathbf{P}, \mathbf{C}, \mathbf{C}', \mathbf{A}, S_0]$ *be a barrier system for* F. *Then every neighborhood* N *of* F *contains a solid torus* S *such that*

(1) S *if* f-*invariant, and* Bd S *is polyhedral relative to* $K(\tilde{U})$,

(2) $\mathbf{B}$ *is a barrier system for* S, *and*

(3) *If* J *is a polygon in a set* Bd $S \cap C_j'$, *and* J *bounds a* 2-*cell in* Bd S, *then* J *bounds a* 2-*cell in* Bd $S \cap \bigcap_r f^r(\text{Int } C_j)$.

PROOF. First take S as a solid torus $\tilde{S}_X$ as in Theorem 5.3. Thus (1) is satisfied. If S lies in a sufficiently small neighborhood of F, then $S \subset N$, and conditions (1), (2), and (3) of Definition 6.2 are satisfied. To get (4), we move Pr Bd S slightly, so that no edge of Pr Bd S intersects any set Pr e, where e is an edge of a set Bd $C_j' - F$; we then lift by Pr^{-1} to get a new S which satisfies (1)–(4) of Definition 6.2.

It remains to show that S can be chosen so that (3) of Theorem 6.5 is also satisfied. For each j, let P_j^- and P_j^+ be the points of $F \cap$ Bd C_j', such that P_{j-1}, $P_{j-1}^{\pm}$, P_j^-, P_j, P_j^+ appear in the stated cyclic order on F. The points P_j^-, P_j^+ lie in the interiors of 2-cells D_j^-, D_j^+, lying in Bd C_j', such that

$$\bigcup_i f^i(D_j^-), \quad \bigcup_i f^i(D_j^+) \subset N$$

and

$$\bigcup_i f^i(D_j^-) \cap \bigcup_k A_k = \bigcup_i f^i(D_j^+) \cap \bigcup_k A_k = \varnothing.$$

(Any sufficiently small 2-cell neighborhoods of P_j^- and P_j^+ in Bd C_j' will do.) Note that under the conditions for a barrier system for F, we automatically have

$$f^i(D_j^-), f^i(D_j^+) \subset C_j \subset S_0,$$

and $f^i(D_j^-) \cap A_k = f^i(D_j^+) \cap A_k = \varnothing$ for $j \neq k$. Then we choose S, subject to all the above conditions, such that

$$\text{Bd } S \cap \text{Bd } C_j' \subset \text{Int } D_j^- \cup \text{Int } D_j^+$$

for each j.

For each j, consider the set

$$Y = \text{Pr}\left[\text{Bd } S \cap \bigcup_r f^r(C_j')\right].$$

This is a finite polyhedron (relative to $K(U)$) in $\Omega - \text{Pr } F$. Let X_j be a compact polyhedral 2-manifold with boundary which forms a neighborhood of Y in Pr Bd S, sufficiently small so that

$$X_j \subset \text{Pr} \bigcap_r f^r(\text{Int } C_j),$$

and let

$$\tilde{X}_j = \text{Pr}^{-1} X_j.$$

Then $\tilde{X}_j$ is also a compact polyhedral 2-manifold with boundary. Let p_j^1 be the 1-dimensional Betti number of X_j (with integers modulo 2 as coefficients). Since each p_j^1 is finite, we may assume, subject to all the above conditions, that S and X_j are chosen so as to minimize Σp_j^1. On this basis we shall show that S satisfies condition (3) of Theorem 6.5.

Suppose that (3) does not hold, and let J be a polygon, with $J \subset \text{Bd } S \cap C_j'$, such that J bounds a 2-cell D in Bd S, but J does not bound a 2-cell in

$$\text{Bd } S \cap \bigcap_r f^r(\text{Int } C_j).$$

By general position, $D \cap \text{Bd } C_j'$ is a finite union of disjoint polygons J_i. Since each J_i bounds a 2-cell in Bd S, it follows that no J_i separates P_j^- from P_j^+ in Bd C_j'. Therefore each J_i bounds a 2-cell D_i in $\text{Bd } C_j' - F$; and since J_i lies in D_j^- or D_j^+, so also does D_i. Evidently $J_i \subset \tilde{X}_j = \text{Pr}^{-1} X_j$. If each J_i bounds a 2-cell in $\tilde{X}_j$, then it follows that

$$D \subset \tilde{X}_j \subset \text{Bd } S \cap \bigcap_r f^r(\text{Int } C_j),$$

which contradicts our assumption for J. Therefore some J_i does not bound a 2-cell in X_j; and we may suppose that J_i is inmost in Bd C_j', in the sense that

$$D_i \subset \text{Bd } C_j' - F$$

contains no other polygon J_k which satisfies the conditions for J_i. Thus every J_k that lies in Int D_i bounds a 2-cell in

$$\text{Bd } S \cap \bigcap_r f^r(\text{Int } C_j).$$

It follows that J_i bounds a 2-cell D_i', lying in $\bigcap_r f^r(\text{Int } C_j)$, such that Int D_i' lies in either $S_0 - \text{Int } S$ or $S - F$. D_i' can be chosen so as to lie in the union of Int D_j^- ∪ Int D_j^+ and any given neighborhood of Bd S. Therefore we may assume that

$$D_i' \cap \text{Bd } C_j' \subset \text{Int } D_j^- \cup \text{Int } D_j^+,$$

and that

$$\bigcup_r f^r(D_i') \cap A_j = \varnothing.$$

(See the proof of Lemma 6.4.1 for the "forcing off" process that is needed here.)

Now Pr$|J_i$ is a loop L in Int X_j = Int Pr $\tilde{X}_j$, and L is contractible in one of the sets

$$\text{Pr}\Big[(S_0 - \text{Int } S) \cap \bigcap_r f^r(\text{Int } C_j)\Big], \qquad \text{Pr}\Big[(S - F) \cap \bigcap_r f^r(\text{Int } C_j)\Big].$$

If L were contractible in X_j, then J_i would be contractible in $\tilde{X}_j$, which is false. Therefore L is not contractible in X_j. We now apply Theorem 3.5 (the Loop Theorem, second form) to the 3-manifold

$$M^3 = \text{Pr}\Big[(\text{Int } S_0 - \text{Bd } S) \cap \bigcap_r f^r(\text{Int } C_j)\Big] \cup \text{Int } X_j - \text{Pr } F$$

and the 2-manifold Int X_j, which forms a closed set in the space M^3. By Theorem 3.5 it follows that there is a 2-cell Δ, polyhedral relative to $K(U)$, lying in

$$\text{Pr}\Big[\bigcap_r f^r(\text{Int } C_j)\Big] - \text{Pr } F,$$

such that

$$\Delta \cap \text{Pr Bd } S = \text{Bd } \Delta \subset \text{Int } X_j,$$

and such that Bd Δ is not contractible in Int X_j, and hence not in X_j. Since Δ can be chosen in an arbitrarily small neighborhood of Pr D_i', and $\bigcup_r f^r(D_i') \cap A_j = \varnothing$, we may assume that $\Delta \cap \text{Pr } A_j = \varnothing$. It follows that $\text{Pr}^{-1}\Delta \cap A_j = \varnothing$.

Now Bd Δ can be lifted, so as to give polygons J_k' such that Pr J_k' = Bd Δ. Since Pr is a local homeomorphism, it follows that $\text{Pr}^{-1}\text{Bd } \Delta$ is the union of exactly n such polygons J_k' $(1 \leqslant k \leqslant n)$. We choose the notation so that $J_{k+1}' = f(J_k')$, with subscripts modulo n. The polygons J_k' cannot be latitudinal in S, because they bound 2-cells in $S^3 - F$. By Theorem 6.4 it follows that each J_k' bounds a 2-cell d_k in Bd S; and we have $d_{k+1} = f(d_k)$. Different sets J_k' are disjoint. It follows that if two different sets d_k intersect, then one of them lies in the interior of the other, which is impossible, because f is periodic. Therefore different sets d_k are disjoint.

Now $\mathrm{Pr}^{-1}\Delta$ is the union of exactly n disjoint 2-cells Δ_k, with

$$\mathrm{Bd}\,\Delta_k = J'_k, \quad f(\Delta_k) = \Delta_{k+1}.$$

In Bd S, we replace each d_k by the corresponding Δ_k. This gives a torus T'. Evidently T' separates F from Bd S_0 in M. Therefore, by Theorem 4.10, T' is the boundary of a solid torus S', such that S' is f-invariant. By arbitrarily small changes in Δ, preserving all the stated properties of Δ, we can produce a situation in which Bd S' is in general position relative to each set Bd C'_i ($C'_i \in \mathbf{C}$).

Now define a new set X'_j by deleting $X_j \cap \mathrm{Pr}\, d_k$ from X_j and adding Δ_k. Then S', X'_j, and $\tilde{X}'_j = \mathrm{Pr}^{-1}X'_j$ satisfy all the conditions for S, X_j, and $\tilde{X}_j$. But all this is impossible: it preserves the stated properties of S, and reduces one of the numbers p_j^1, without increasing any of the others. This contradicts the hypothesis that S was chosen so as to minimize $\Sigma_j p_j^1$.

7. Simplifications of f and Ω. Here we use the apparatus of §6.

THEOREM 7.1. *Let* $\mathbf{B} = [\mathbf{P}, \mathbf{C}, \mathbf{C}', \mathbf{A}, S_0]$ *be a barrier system for* S, *and suppose that* S *is* f-*invariant. Then every set* Bd $S \cap C_j$ *contains a polygon* J *such that* (1) J *is latitudinal and* (2) J *is* f-*invariant.*

PROOF. By Theorem 6.3 we know that Bd $S \cap$ Bd C'_j contains a polygon J_0 which is latitudinal in S. And by definition of a barrier system we have

$$\bigcup_r f^r(C'_j) \subset \bigcap_r f^r(\mathrm{Int}\, C_j).$$

It follows that

$$\bigcup_r f^r(J_0) \subset \mathrm{Int}\, C_j.$$

Now Pr J_0 is a polyhedron relative to $K(U)$. Let $K'(U)$ be a subdivision of $K(U)$ in which Pr J_0 forms a subcomplex, and let N be a regular neighborhood of Pr J_0 in Pr Bd S, defined relative to a subdivision of $K'(U)$. Then N is a 2-manifold with boundary. (We need not assert that N is connected.) Therefore so also is $\tilde{N} = \mathrm{Pr}^{-1}N$; and obviously $\tilde{N}$ is f-invariant. We take N as a sufficiently small neighborhood of Pr J_0 so that $\tilde{N} \subset \mathrm{Int}\, C_j$.

Since every other set Bd $S \cap C'_k$ contains a latitudinal polygon, it follows as in the proof of Theorem 6.4 that every polygon in $\tilde{N}$ is latitudinal in S or bounds a 2-cell in Bd S. To $\tilde{N}$ we add all 2-cells $d \subset$ Bd S such that Bd d is a component of Bd $\tilde{N}$. This gives a 2-manifold W with boundary, such that

$$\mathrm{Bd}\, W \subset \mathrm{Bd}\, \tilde{N} \subset \mathrm{Int}\, C_j,$$

and such that every component of Bd W is latitudinal in S. It follows that every component of W is an annulus. And obviously W and Bd W are f-invariant.

Let J_1 be a latitudinal polygon in a set $\text{Bd}\, S \cap \text{Bd}\, C'_k$ $(k \neq j)$. Then $\text{Bd}\, S - J_1$ is homeomorphic to the interior of an annulus, and no component of Bd M bounds a 2-cell in $\text{Bd}\, S - J_1$. It follows that there is an annulus $A \subset \text{Bd}\, S - J_1$ such that $W \subset A$, $\text{Bd}\, A \subset \text{Bd}\, W$. Here A is the union of W and a collection of annuli a_i whose boundaries lie in W and whose interiors are disjoint from W. This property of a_i is f-invariant. Therefore A and Bd A are f-invariant. Since f preserves orientation, f cannot interchange the components of Bd A. Therefore each component of Bd A is f-invariant. Since

$$\text{Bd}\, A \subset \text{Bd}\, W \subset \text{Bd}\, \tilde{N} \subset \text{Int}\, C_j,$$

the theorem follows.

THEOREM 7.2. *Under the hypothesis of Theorem* 7.1, *let* J_j *be an f-invariant latitudinal polygon in S, lying in a set* C_j. *Then there is a triangulation* $L(M)$ *of M, and a homeomorphism* $f'\colon M \leftrightarrow M$, *with fixed-point set* $F' \subset \text{Int}\, S$, *such that*

(1) F', *S, and* J_j *form subcomplexes of* $L(M)$,

(2) F' *is a polygon,*

(3) F' *has period n,*

(4) f' *is simplicial relative to* $L(M)$,

(5) $\text{Cl}(M - S)$ *forms a subcomplex* L' *of* $L(M)$, *and* L' *is a subcomplex of a subdivision of* $K(\tilde{U})$,

(6) $f'|\text{Cl}(M - S) = f|\text{Cl}(M - S)$, *and*

(7) *The orbit-space* Ω' *of* f' *is a* 3-*manifold, and the sets* $\text{Pr}'\,\sigma$ (*where* $\sigma \in L(M)$ *and* Pr′ *is the projection* $M \twoheadrightarrow \Omega'$) *form a triangulation of* Ω'.

PROOF. By Theorem 7.1 we know that there is another f-invariant latitudinal polygon J_k in S, lying in some set $C_k (k \neq j)$, so that J_j and J_k are disjoint. Thus J_j and J_k bound tame 2-cells D_j, D_k in S, such that

$$D_p \cap \text{Bd}\, S = \text{Bd}\, D_p \qquad (p = j, k).$$

Since the sets Int D_p are tame, it follows by Theorem 3.4 that we may choose them so that the sets Int D_p are polyhedra (not necessarily finite) relative to $K(M)$. By standard methods of "cutting and pasting," we can arrange so that $D_j \cap D_k = \varnothing$.

Now J_j and J_k decompose Bd S into two annuli a_1, a_2. All the sets D_j, D_k, a_1, a_2 are tame. By Theorem 3.4, together with Lemma (2.1) of **[M₈]**, it follows that each set $D_j \cup D_k \cup a_q$ is tame, and therefore bounds a 3-cell C_q^3. Thus

$$S = C_1^3 \cup C_2^3, \qquad C_1^3 \cap C_2^3 = D_j \cup D_k.$$

Let $K'(\tilde{U})$ be a subdivision of $K(\tilde{U})$, such that Bd S and the sets J_p form subcomplexes of $K'(\tilde{U})$. Each simplex σ of $K'(\tilde{U})$ that lies in $\text{Cl}(M - S)$ will be a simplex of $L(M)$.

We form the rest of $L(M)$ as follows. First, we triangulate each D_p so that the triangulation forms the join of a point Q_p of Int D_p and the subcomplex of $K'(\tilde{U})$ formed by J_p. (This triangulation need not be rectilinear, or even polyhedral.) Now each set Bd C_q^3 is already triangulated. We triangulate each C_q^3 in such a way that it forms the join of an interior point R_q of C_q^3 with the given triangulation of Bd C_q^3. This gives $L(M)$.

It is now easy to define f' as an extension of $f|\text{Cl}(M - S)$. First define $f'(Q_p) = Q_p$, and extend f' simplicially to the simplices $Q_p vw$ of D_p. Define $f'(R_q) = R_q$, and extend f' simplicially to the simplices $R_q \sigma^2$ (σ^2 in Bd C_q^3).

$L(M)$ and f' now satisfy (1)–(6). (F' is the union of the four edges $Q_p R_q$ of $L(M)$.) To verify (7) we merely need to show that for each vertex v of F', with closed star St v in $L(M)$, the homeomorphism $f'|$St v is like a periodic rotation of a round ball. This is straightforward.

8. Invariant barrier annuli. If S and $\mathbf{B}$ satisfy all three of the conditions of Theorem 6.5, then we shall say that S *has no cellular oscillations in* $\mathbf{B}$.

THEOREM 8.1. *Let* $\mathbf{B}$ *be a barrier system for* F. *Let* S_1 *and* S_2 *be* f*-invariant solid tori, such that* $S_2 \subset$ Int S_1; *and suppose that for* $i = 1, 2$, S_i *has no cellular oscillations in* $\mathbf{B}$. *If* S_1 *lies in a sufficiently small neighborhood of* F, *then for each* j *there is an annulus* A_j' *such that*

(1) A_j' *is* f*-invariant*,

(2) $A_j' \subset S_1 -$ Int S_2,

(3) $A_j' \cap$ (Bd $S_1 \cup$ Bd S_2) = Bd A_j', *and this set is the union of a latitudinal polygon in* Bd S_1 *and a latitudinal polygon in* Bd S_2,

(4) Bd $A_j' \subset$ Int C_j',

(5) A_j' *intersects a set* C_k *only if* $k = j$ *or* $k = j \pm 1$, *and*

(6) A_j' *is a polyhedron relative to* $K(\tilde{U})$.

PROOF. Take a fixed j, and let d_j be as in the discussion just before the definition of a barrier system for F, in §6. Take S_1 in a sufficiently small neighborhood of F so that

$$\bigcup_r f^r(S_1 \cap d_j) \subset \bigcap_r f^r(\text{Int } C_j').$$

By small changes in d_j (preserving all conditions stated so far) we get d_j into general position relative to Bd S_1 and Bd S_2. By hypothesis for the sets S_i we know that every component J of $d_j \cap$ Bd S_i ($i = 1, 2$) is latitudinal in S_i or bounds a 2-cell in Bd S_i; and in the latter case J bounds a 2-cell in Bd $S_i \cap \bigcap_r f^r(\text{Int } C_j)$.

LEMMA 8.1.1. *For* $i = 1, 2$ *let* M_i^2 *be the component of* $d_j \cap S_i$ *that contains* $P_j = d_j \cap F$. *Then exactly one component* J_i *of* Bd M_i^2 *is latitudinal in* S_i; *the other components of* Bd M_i^2 *bound* 2*-cells in* Bd S_i; *and* J_2 *separates* P_j *from* J_1 *in* d_j.

PROOF. By general position, M_i^2 is a 2-manifold with boundary. Let J_i be the frontier (in d_j) of the component of $d_j - M_i^2$ that contains Bd d_j. Then J_i is a polygon, because M_i^2 is connected. Every other component J of Bd M_i^2 bounds a 2-cell in $d_j - P_j$, and so cannot be latitudinal in S_i. Therefore each such J bounds a 2-cell in Bd S_i. It follows, by a standard "forcing off" process, that J_i is latitudinal in S_i. Since $M_2^2 \subset \text{Int } M_1^2$, J_2 lies in the interior of the 2-cell in d_j bounded by J_1, and the lemma follows.

Note that J_1 is the inmost (in d_j) of the polygons in $d_j \cap \text{Bd } S_1$ that are latitudinal in S_1, and J_2 is the outermost (in d_j) of the latitudinal polygons in $d_j \cap \text{Bd } S_2$ that lie in the 2-cell in d_j bounded by J_1.

LEMMA 8.1.2. *For $i = 1, 2$ there is an f-invariant polyhedral annulus a_i in* Bd S_i *such that*

$$J_i \subset \text{Int } a_i, \qquad \text{Bd } a_i \subset \bigcap_r f^r(\text{Int } C_j'), \qquad a_i \subset \bigcap_r f^r(\text{Int } C_j).$$

PROOF OF LEMMA. Consider the 1-dimensional polyhedron

$$\text{Pr } J_i \subset \text{Pr Bd } S_i.$$

Let N be a small regular neighborhood of Pr J_i in Pr Bd S_i, and let $\tilde{N}$ be the *component* of $\text{Pr}^{-1}N$ that contains J_i. Then $\tilde{N}$ is a connected polyhedral 2-manifold with boundary in Bd S_i, and $J_i \subset \text{Int } \tilde{N}$. Since

$$\bigcup_r f^r(J_1) \subset \bigcap_r f^r(\text{Int } C_j'),$$

we can choose N so that $\tilde{N} \subset \bigcap_r f^r(\text{Int } C_j')$. If J is a component of Bd $\tilde{N}$ which bounds a 2-cell $D_J \subset \text{Bd } S_i$, then $D_J \subset \bigcap_r f^r(\text{Int } C_j)$. We add all such 2-cells to $\tilde{N}$, getting an f-invariant 2-manifold a_i with boundary, such that $J_i \subset \text{Int } a_i$ and such that no component of Bd a_i bounds a 2-cell in Bd S_i. It follows that each component of Bd a_i is latitudinal in S_i. Therefore Bd a_i has exactly two components, and a_i is an annulus, satisfying all the conditions of the lemma. (Hereafter, the notations N, $\tilde{N}$ will *not* refer to the above proof.)

Now $J_1 \cup J_2$ is the boundary of an annulus $B_1 \subset d_j - P_j$. The intersections Int $B_1 \cap \text{Bd } S_i$ may not be empty, but each of them is a finite union of disjoint polygons, each of which bounds a 2-cell in Bd S_i, and hence in

$$\text{Bd } S_i \cap \bigcap_r f^r(\text{Int } C_j).$$

By a standard "forcing off" process, we get an annulus B_2, polyhedral relative to $K(\tilde{U})$, such that

$$B_2 \subset \bigcap_r f^r(\text{Int } C_j), \qquad \text{Bd } B_2 = J_1 \cup J_2 \subset \text{Int } a_1 \cup \text{Int } a_2,$$

$$\text{Int } B_2 \subset \text{Int } S_1 - S_2.$$

Adding to B_2 a 2-cell in S_2, with J_2 as a boundary, and with its interior in

Int S_2, we get a 2-cell $D \subset S_1$, with

$$\operatorname{Int} D \subset \operatorname{Int} S_1, \qquad D \cap \operatorname{Bd} S_i \subset \operatorname{Int} a_i \subset \bigcap_r f^r(\operatorname{Int} C_j) \qquad (i = 1, 2),$$

and

$$D - \operatorname{Int} S_2 \subset \bigcap_r f^r(\operatorname{Int} C_j),$$

so that

$$\operatorname{Pr} D - \operatorname{Pr} \operatorname{Int} S_2 \subset \operatorname{Pr} \bigcap_r f^r(\operatorname{Int} C_j).$$

We now use S_2 as the S of the hypothesis of Theorem 7.2. Consider the f', Ω', Pr′: $M \to \Omega'$ given by the conclusion of Theorem 7.2. Thus Pr′ S_2 is a PL 3-manifold with boundary, and so also is Pr′ S_1. Note that Pr′(M − Int S_2) = Pr(M − Int S_2); in fact, Pr′|(M − Int S_2) = Pr|(M − Int S_2). Let W be a closed neighborhood of Pr($D \cup \operatorname{Int} a_1 \cup \operatorname{Int} a_2$), in Pr′($S_1$) regarded as a space, such that

(1) W is a polyhedral 3-manifold with boundary,

(2) $W \cap \operatorname{Pr}' \operatorname{Bd} S_i = \operatorname{Pr} a_i$ $(i = 1, 2)$, and

(3) $W - \operatorname{Pr} \operatorname{Int} S_2 \subset \operatorname{Pr} \bigcap_r f^r(\operatorname{Int} C_j)$.

Let $N = \operatorname{Pr} a_1$, and let $M^3 = \operatorname{Int} W \cup \operatorname{Int} N$. Then M^3 is a 3-manifold with boundary, and Bd M^3 = Int N. Evidently there is a loop in Int N (namely, Pr|J_1) which is contractible in M^3 but not in Int N. By the Loop Theorem it follows that there is a PL 2-cell Δ_1 in M^3, with

$$\operatorname{Bd} \Delta_1 = \Delta_1 \cup \operatorname{Int} N,$$

such that Bd Δ_1 is not contractible in Int N. By an isotopy we can move Δ_1 so as to get a PL 2-cell Δ such that

$$\Delta \subset W, \quad \operatorname{Bd} \Delta = \Delta \cap \operatorname{Bd} W \subset \operatorname{Bd} N,$$

so that Bd Δ is not contractible in N.

If Bd Δ were contractible in Pr′ Bd S_1, then each component of the annulus N = Pr′ a_1 would be contractible in Pr′ Bd S_1; each component of Bd a_1 would be contractible in Bd S_1, and so J_1 would be contractible in Bd S_1, which is false. Thus we have the following:

(i) $\Delta \cap \operatorname{Bd} \operatorname{Pr}' S_1 = \operatorname{Bd} \Delta \subset \operatorname{Bd} \operatorname{Pr}' a_1$,

(ii) $\Delta \cap \operatorname{Bd} \operatorname{Pr}' S_2 \subset \operatorname{Pr}' a_2$,

(iii) $\operatorname{Int} \Delta \subset \operatorname{Pr}' \operatorname{Int} S_1$,

(iv) Bd Δ is not contractible in Pr′ Bd S_1, and

(v) $(\Delta - \operatorname{Pr}' \operatorname{Int} S_2) - \operatorname{Pr}' \bigcap_r f^r(\operatorname{Int} C_j)$ lies in a finite union of disjoint polyhedral 2-cells, disjoint from Pr′ Bd S_1, from Pr′ Bd S_2, and from one another.

(In fact, at the outset we have

(v′) $\Delta - \operatorname{Pr}' \operatorname{Int} S_2 \subset \operatorname{Pr}' \bigcap_r f^r(\operatorname{Int} C_j)$.

But (v) needs to be stated in such a form that it would be preserved by certain hypothetical operations to be discussed presently.)

We may also assume that

(vi) Δ is in general position relative to Pr′ Bd S_2, and Pr'^{-1} Int Δ is in general position relative to each set Bd C'_k.

(Note that (i) implies that Bd $\Delta \cap$ Bd $C'_k = \varnothing$ for each k.)

Finally, we may assume that

(vii) Subject to conditions (i)–(vi), Δ is chosen so as to minimize the number p of components of $\Delta \cap$ Pr′ Bd S_2.

LEMMA 8.1.3. *No component J of $\Delta \cap$ Pr′ Bd S_2 bounds a 2-cell in* Pr′ Bd S_2.

PROOF OF LEMMA. Suppose that

$$J = \mathrm{Bd}\, D_J, \qquad D_J \subset \mathrm{Pr}'\, \mathrm{Bd}\, S_2.$$

We may suppose that J is inmost in Pr′ Bd S_2, in the sense that Int D_J contains no component of $\Delta \cap$ Pr′ Bd S_2. Now J bounds a 2-cell $D'_J \subset \Delta$. We substitute D_J for D'_J in Δ, and force the resulting 2-cell slightly off Pr′ Bd S_2 in the neighborhood of D_J. Minor adjustments restore condition (vi). All this is impossible, because it preserves (i)–(vi) and reduces the p of (vii).

Now let D' be a 2-cell which is a lifting of Δ, so that Pr′ $D' = \Delta$. Let A'_j be the component of $D' -$ Int S_2 that contains Bd D'.

LEMMA 8.1.4. *Every component of* Bd $A'_j -$ Bd S_1 *is latitudinal in* S_2.

PROOF OF LEMMA. Since each such component J is disjoint from a latitudinal polygon (lying in some set Bd $S_2 \cap$ Bd C'_r $(r \neq j)$), it follows that J is latitudinal in S_2 or bounds a 2-cell in Bd S_2. If the latter held, then Pr′ J would bound a 2-cell in Pr′ Bd S_2, which is false.

LEMMA 8.1.5. *Every component of* Bd $A'_j -$ Bd S_1 *is f-invariant. So also is* A'_j.

PROOF OF LEMMA. Evidently the entire set $\mathrm{Pr}'^{-1}\, \Delta$ is f'-invariant. Applying Theorem 7.1 in some set C'_r $(r \neq j)$, we express Bd S_2 as the union of two f-invariant annuli, whose intersection is the boundary of each, such that Bd $S_2 \cap \mathrm{Pr}'^{-1}\, \Delta$ lies in the union of their interiors. Since f is periodic, and preserves orientation, the components of Bd $A'_j \cap$ Bd S_2 cannot be permuted by f' (which agrees with f except in Int S_2). Therefore each of these components is f-invariant. Therefore so also is A'_j.

LEMMA 8.1.6. *A'_j is an annulus.*

PROOF OF LEMMA. We need to show that Bd $A'_j \cap$ Bd S_2 is connected. Now A'_j can be regarded as a complex (that is, a subcomplex of a subdivision of $K(\tilde{U})$). Assign orientations to the simplices of A'_j, in such a way that by assigning coefficient 1 to each 2-simplex of A_j' we get a 2-chain C^2, with integers as coefficients, such that

$$\partial C^2 = Z^1 = Z_1^1 + Z_2^1,$$

where Z_i^1 is a 1-cycle on Bd S_i $(i = 1, 2)$. Now

$$Z_2^1 = \sum_s Y_s^1,$$

where each Y_s^1 has constant coefficient 1 or -1 on its polygonal carrier $|Y_s^1|$, which is a component of $A_j' \cap \text{Bd } S_2$.

Suppose that there is more than one such component $|Y_s^1|$. Since all of them are latitudinal in S_2, we have $Y_r^1 \sim \pm Y_s^1$ for every r and s. Therefore there are two cycles Y_r^1, say, Y_1^1 and Y_2^1, such that $Y_1^1 + Y_2^1 \sim 0$ on Bd S_2. Let b be a broken line in A_j', from a point of $|Y_1^1|$ to a point of $|Y_2^1|$, intersecting Bd A_j' only at its end-points, and let N be a small regular neighborhood of $|Y_1^1| \cup |Y_2^1| \cup b$ in A_j'. Then Bd N is a polygon J, and J carries a 1-cycle Z_J such that $Z_J \sim Y_1^1 + Y_2^1 \sim 0$ on $S_1 - \text{Int } S_2$. Therefore $Z_J \sim 0$ on Int $S_1 - \text{Int } S_2$. Evidently J is not contractible in A_j': J separates A_j' into two connected sets each of which contains a component of Bd A_j'.

By Theorem 4.10 we know that Int $S_1 - S_2$ is the interior of a toroidal shell, and

$$\pi(\text{Int } S_1 - S_2) \approx H_1(\text{Int } S_1 - S_2, \mathbf{Z}) \approx \mathbf{Z} + \mathbf{Z}.$$

Thus the canonical homomorphism

$$\pi(\text{Int } S_1 - S_2) \to H_1(\text{Int } S_1 - S_2, \mathbf{Z})$$

is an isomorphism. Since $Z_J \sim 0$ on Int $S_1 - S_2$, it follows that any loop L: $S^1 \to J$ that traverses J exactly once is contractible in Int $S_1 - S_2$. Since the homomorphism

$$(\text{Pr}|\text{Int } A_j')^*: \pi(A_j') \to \pi(\text{Pr } A_j')$$

is injective, it follows that Pr L: $S^1 \to \text{Pr Int } A_j'$ is not contractible in Pr Int A_j'. We now apply the Loop Theorem (second form) to the 3-manifold Pr(Int $S_1 - S_2$) and the 2-manifold Pr Int A_j'. This gives a PL 2-cell

$$\Delta_1 \subset \text{Pr}(\text{Int } S_1 - S_2),$$

with

$$\Delta_1 \cap \text{Pr } A_j' = \text{Bd } \Delta_1 \subset \text{Pr Int } A_j',$$

such that Bd Δ_1 is not contractible in Pr A_j'. Obviously Δ_1 can be slid off all the 2-cells mentioned in (v) above, and so we may assume that

$$\text{Bd } \Delta_1 \subset \text{Pr} \bigcap_r f^r(\text{Int } C_j).$$

Now Bd Δ_1 bounds a 2-cell Δ_2 in Δ, and Int Δ_2 must contain at least one component of $\Delta \cap \operatorname{Pr} \operatorname{Bd} S_2$. We replace Δ_2 by Δ_1 in Δ, getting a 2-cell Δ'. We then move Int Δ' slightly, so that $\operatorname{Pr}'^{-1} \operatorname{Int} \Delta'$ is in general position relative to each set Bd C_j'.

But all this is impossible: it preserves conditions (i)–(v), and reduces the number of components of $\Delta \cap \operatorname{Pr}' \operatorname{Bd} S_2$. The lemma follows.

Thus A_j' is an annulus. By conditions (i) and (ii) for Δ we have

$$\operatorname{Pr}' \operatorname{Bd} A_j' \subset \operatorname{Pr}' \operatorname{Bd} a_1 \cup \operatorname{Pr}' a_2.$$

Now A_j' satisfies (1)–(3) and (6) of Theorem 8.1. To get an A_j' satisfying (4), we move the polygon $\Delta \cap \operatorname{Pr}' \operatorname{Bd} S_2$ onto a component of $\operatorname{Pr}' \operatorname{Bd} a_2$, by an isotopy $\Omega' \leftrightarrow \Omega'$, $\operatorname{Pr}' \operatorname{Bd} S_2 \leftrightarrow \operatorname{Pr}' \operatorname{Bd} S_2$ which differs from the identity only in a small neighborhood of $\operatorname{Pr}' a_2$. This gives a new Δ, satisfying (i)–(iv). Since $a_2 \subset \bigcap_r f^r(\operatorname{Int} C_j)$, (v) is also satisfied. The new Δ can be moved into general position so as to satisfy (vi). Lifting, we get a new A_j' such that

$$\operatorname{Bd} A_j' \subset \operatorname{Bd} a_1 \cup \operatorname{Bd} a_2 \subset \operatorname{Int} C_j'.$$

Thus the new A_j' satisfies (1)–(4) and (6).

It remains only to show that A_j' can be chosen so as to satisfy (5). The proof is very similar to that of Theorem 6.5, as follows. Consider the sets

$$A_j' \cap \operatorname{Bd} C_{j-1}', \qquad A_j' \cap \operatorname{Bd} C_{j+1}'.$$

Each of these is a finite union of disjoint polygons J. If each of these bounds a 2-cell in $A_j' \cap \bigcap_r f^r(\operatorname{Int} C_{j-1})$ or $A_j' \cap \bigcap_r f^r(\operatorname{Int} C_{j+1})$, then A_j' satisfies (5). Thus, if (5) does not hold, then there is such a J, lying, say, in $A_j' \cap \operatorname{Bd} C_{j-1}'$, such that J does not bound a 2-cell in $A_j' \cap \bigcap_r f^r(\operatorname{Int} C_{j-1})$.

By (v), each such J bounds a 2-cell in A_j'. Now J bounds a 2-cell D_J in $\operatorname{Bd} C_{j-1}' - A_{j-1}$. If D_J intersects F, then every polygon in $\operatorname{Bd} S_1 \cap \operatorname{Bd} C_{j-1}'$ bounds a 2-cell in S_1, disjoint from F; and this is impossible, because some polygon in $\operatorname{Bd} S_1 \cap \operatorname{Bd} C_{j-1}'$ is latitudinal in S_1. Therefore

$$D_J \subset (\operatorname{Bd} C_{j-1}' - A_{j-1}) - F.$$

We may suppose that D_J is irreducible with respect to its stated properties, so that if J' is a polygon in $A_j' \cap \operatorname{Int} D_J$, then J' bounds a 2-cell in $A_j' \cap \bigcap_r f^r(\operatorname{Int} C_{j-1})$. No component J' of

$$(\operatorname{Bd} S_1 \cup \operatorname{Bd} S_2) \cap \operatorname{Bd} C_{j-1}'$$

that lies in D_J can be latitudinal in S_1 or S_2; and by Theorem 6.4 it follows that every such J' bounds a 2-cell in Bd S_1 (or Bd S_2). By the hypothesis of Theorem 8.1., J' bounds a 2-cell in

$$\operatorname{Bd} S_i \cap \bigcap_r f^r(\operatorname{Int} C_{j-1}) \qquad (i = 1 \text{ or } i = 2).$$

By the usual forcing off process we conclude that J bounds a 2-cell D_J' in

$$(\text{Int } S_1 - S_2) \cap \bigcap_r f^r(\text{Int } C_{j-1}),$$

such that $D'_j \cap A'_j = J$.

All this can be ruled out by a minimality assumption similar to the one used in the proof of Theorem 6.5. For each j, consider the set

$$Y = \text{Pr}\left[A'_j \cap \bigcup_r f^r(C'_{j-1})\right].$$

This is a finite polyhedron (relative to $K(U)$). Let X_{j-1} be a polyhedral 2-manifold with boundary which forms a neighborhood of Y in $\text{Pr } A'_j$, such that

$$X_{j-1} \subset \text{Pr} \bigcap_r f^r(\text{Int } C_{j-1}).$$

Then X_{j-1} and $\tilde{X}_{j-1} = \text{Pr}^{-1}X_{j-1}$ are 2-manifolds with boundary. Let $H_1(X_{j-1})$ be the 1-dimensional homology group of X_{j-1}, with coefficients in the additive group $\mathbf{Z}_2$ of integers modulo 2.

Let J and D'_j be as in the preceding discussion. Let V be the component of

$$\left[(\text{Int } S_1 - S_2) \cap \bigcap_r f^r(\text{Int } C_{j-1})\right] - A'_j$$

that contains $\text{Int } D'_j$. Since $\text{Int } S_1$, S_2, $\bigcap_r f^r(\text{Int } C_{j-1})$, and A'_j are all f-invariant, so also is V. Let

$$W = \text{Pr } V \cup \text{Int } X_{j-1}.$$

Then W is a PL 3-manifold with boundary, with $\text{Bd } W = \text{Int } X_{j-1}$, and $\text{Pr}|J$ is a loop in $\text{Bd } W$, contractible in W but not in $\text{Bd } W$. By the Loop Theorem it follows that there is a polyhedral 2-cell $\Delta \subset W$ such that $\text{Bd } \Delta = \Delta \cap \text{Bd } W$ and $\text{Bd } \Delta$ is not contractible in $\text{Bd } W$. As in the proof of Theorem 6.5, $\text{Bd } \Delta$ can be lifted, to give polygons J'_k $(1 < k < n)$ in A'_j, such that $\text{Pr } J'_k = \text{Bd } \Delta$. As before, these polygons are disjoint, and bound disjoint 2-cells d_k in A'_j. Similarly, $\text{Pr}^{-1} \Delta$ is the union of n disjoint 2-cells Δ_k, with $\text{Bd } \Delta_k = J'_k$; and since $\Delta \subset W$ we have

$$\Delta_k \subset (\text{Int } S_1 - S_2) \cap \bigcap_r f^r(\text{Int } C_{j-1}).$$

In A'_j, we replace each d_k by the corresponding Δ_k, thus getting an f-invariant annulus A''_j. By arbitrarily small changes in Δ, we can arrange so that A''_j is in general position relative to $\text{Bd } C'_{j-1}$.

Now define a new set X'_{j-1} by deleting $X_{j-1} \cap \text{Pr } d_k$ from X_{j-1} and replacing it with Δ. Then A''_j and X'_{j-1} have all the stated properties of A'_j and X_{j-1}: A''_j is an f-invariant annulus, satisfying (1)–(4) and (6) of Theorem 8.1; X'_{j-1} is a polyhedral 2-manifold with boundary, forming a neighborhood of

$$Y' = \Pr\left[A_j'' \cap \bigcup_r f^r(C'_{j-1})\right]$$

in Pr A_j'', and

$$X'_{j-1} \subset \Pr \bigcap_r f^r(\operatorname{Int} C_{j-1}).$$

But the operation $A_j' \to A_j''$, $X_{j-1} \to X'_{j-1}$ reduces the dimension of $H_1(X_{j-1})$ (as a vector space over $\mathbf{Z}_2$), without increasing the dimension of the similarly defined group $H_1(X_{j+1})$. Therefore, to get an A_j' satisfying (1)–(6), it is sufficient to choose an A_j' satisfying (1)–(4) and (6), in such a way as to minimize the sum of the dimensions of $H_1(X_{j-1})$ and $H_1(X_{j+1})$.

9. Eliminating annular oscillations.

THEOREM 9.1. *Let* $\mathbf{B}' = [\mathbf{P}, \mathbf{C}, \mathbf{C}', \mathbf{A}, S_0]$ *be a barrier system for* F, *and suppose that the number of points in* P *is* $4m$ ($m \geqslant 3$). *Let* $\mathbf{B}$ *be the barrier system which uses the same* S_0, *and uses the points* P_{4j} *and the sets* C_{4j}, C'_{4j}, A_{4j}. *Then every neighborhood of* F *contains an f-invariant solid torus* S *such that*

(1) *S and* $\mathbf{B}$ *satisfy all the conditions for* S_1 *and* $\mathbf{B}$ *in Theorem* 8.1, *and*

(2) Bd S *is the union of a collection of annuli* M_j ($1 \leqslant j \leqslant m$), *intersecting only on their boundaries, such that* (2.1) M_j *is f-invariant,* (2.2) Bd M_j *is the union of a latitudinal polygon in* $S \cap \operatorname{Int} C_{4j}$ *and a latitudinal polygon in* $S \cap \operatorname{Int} C_{4j+4}$, *and* (2.3) M_j *intersects* C_{4k} *only if* $k = j$ *or* $k = j + 1$.

PROOF. Let S_1 be a solid torus such that S_1 and $\mathbf{B}'$ satisfy all the conditions for S_1 and $\mathbf{B}$ in Theorem 8.1. Thus if S_2 is a solid torus lying in Int S_1, and S_2 has no cellular oscillations in $\mathbf{B}'$, then the conclusion of Theorem 8.1 holds for S_1, S_2, and $\mathbf{B}'$. In Theorem 8.1 it was merely required that S_1 be a "sufficiently small" neighborhood of F, with no cellular oscillations in $\mathbf{B}$. Thus S_2 and $\mathbf{B}'$ automatically satisfy the conditions for S_1 and $\mathbf{B}$ in Theorem 8.1.

Now F intersects each set Bd C'_{4j+2} in two points P^-_{4j+2}, P^+_{4j+2}, and these points have 2-cell neighborhoods d^-_{4j+2}, d^+_{4j+2} in Bd C'_{4j+2}, disjoint from Bd S_1. By Theorem 6.5 there s a solid torus S such that (i) S has no cellular oscillations in $\mathbf{B}'$, (ii) $S \subset \operatorname{Int} S_1$, and (iii) for each j,

$$S \cap \operatorname{Bd} C'_{4j+2} \subset \operatorname{Int} d^-_{4j+2} \cup \operatorname{Int} d^+_{4j+2}.$$

Now P^-_{4j+2} and P^+_{4j+2} have 2-cell neighborhoods

$$d'^-_{4j+2} \subset \operatorname{Int} d^-_{4j+2}, \qquad d'^+_{4j+2} \subset \operatorname{Int} d^+_{4j+2}$$

in Bd C'_{4j+2}, disjoint from Bd S. Let S_2 be an f-invariant solid torus as in Theorem 8.1, such that

$$S_2 \cap \operatorname{Bd} C'_{4j+2} \subset \operatorname{Int} d'^+_{4j+2} \cup \operatorname{Int} d'^-_{4j+2};$$

and let $\{A'_{4j+2}\}$ be as in the conclusion of Theorem 8.1. (Throughout, we are using $\mathbf{B}'$ as the $\mathbf{B}$ of Theorem 8.1, but using only the resulting annuli of the form A'_{4j+2}.) From (5) of Theorem 8.1 it follows that every two different sets A'_{4j+2} are disjoint, and that none of them intersects any set C_{4k}.

By Theorem 8.1, we have $\text{Bd}\, A'_{4j+2} \subset \text{Int}\, C'_{4j+2}$. The polygons in $\text{Bd}\, S_1 \cap \text{Bd}\, C'_{4j+2}$ which are latitudinal in S_1 decompose $\text{Bd}\, S_1$ into a finite collection of annuli, one of which, $A_{1,j}$, contains

$$\text{Bd}\, A'_{4j+2} \cap \text{Bd}\, S_1 \subset \text{Int}\, C'_{4j+2}$$

in its interior. Similarly, the latitudinal polygons in $\text{Bd}\, S_2 \cap \text{Bd}\, C'_{4j+2}$ decompose $\text{Bd}\, S_2$ into annuli, one of which, $A_{2,j}$, contains the other component of $\text{Bd}\, A'_{4j+2}$ in its interior. The components J of $\text{Bd}\, S_i \cap \text{Bd}\, C'_{4j+2}$ which are not latitudinal in S_i bound 2-cells in $\text{Bd}\, S_i$ (Theorem 6.4). Since S_1 and S_2 have no cellular oscillations in $\mathbf{B}'$, it follows that

$$A_{ij} \subset \bigcap_r f^r(\text{Int}\, C_{4j+2}) \qquad (i = 1, 2).$$

We now form the union L_j of

(1) The annulus A_{4j+2} (as in the definition of a barrier system, just before Theorem 6.1),

(2) An annulus in $A_{1,j}$, bounded by the union of $\text{Bd}\, A'_{4j+2} \cap \text{Bd}\, S_1$ and a latitudinal polygon $J_1 \subset \text{Bd}\, S_1 \cap \text{Bd}\, C'_{4j+2}$,

(3) A'_{4j+2},

(4) An annulus in $A_{2,j}$, bounded by the union of $\text{Bd}\, A'_{4j+2} \cap \text{Bd}\, S_2$ and a latitudinal polygon J_2 in $\text{Bd}\, S_2 \cap \text{Bd}\, C'_{4j+2}$,

(5) An annulus in $\text{Bd}\, C'_{4j+2} - (d^-_{4j+2} \cup d^+_{4j+2})$, bounded by J_1 and a boundary component of A_{4j+2}, and

(6) A 2-cell in d'^+_{4j+2} or d'^-_{4j+2}, bounded by J_2.

LEMMA 9.1.1. *L_j is (the image-set of) a singular 2-cell with no singularities on its boundary.*

To see this, consider the above sets in the order 1, 5, 2, 3, 4, 6.

LEMMA 9.1.2. *$L_j \cap \text{Bd}\, S \subset \text{Int}\, A'_{4j+2}$ for each j.*

PROOF OF LEMMA. $\text{Bd}\, S \cap (1) = \varnothing$, because $\mathbf{B}'$ is a barrier system for S. $\text{Bd}\, S \cap (2) = \varnothing$, because $(2) \subset A_{1,j} \subset \text{Bd}\, S_1$. $\text{Bd}\, S \cap (4) = \varnothing$, because $(4) \subset \text{Bd}\, S_2$. $\text{Bd}\, S \cap (5) = \varnothing$, because $\text{Bd}\, S \cap \text{Bd}\, C'_{4j+2} \subset d^+_{4j+2} \cup d^-_{4j+2}$. $\text{Bd}\, S \cap (6) = \varnothing$, because $\text{Bd}\, S \cap (d'^+_{4j+2} \cup d'^-_{4j+2}) = \varnothing$.

LEMMA 9.1.3. *$L_j \cap C_{4k} = \varnothing$ for each k.*

By construction: recall that $A'_{4j+2} \cap C_{4k} = \varnothing$, and

$$A_{1,j}, A_{2,j} \subset \bigcap_r f^r(\text{Int}\, C_{4j+2}).$$

LEMMA 9.1.4. *Every two different sets C_{4r}, C_{4s} lie in different components of $S_0 - \bigcup_{j=1}^{m} L_j$.*

PROOF OF LEMMA. Suppose not. Then a point of some C_{4r} can be joined to a point of a different set C_{4s} by a broken line b in $S_0 - \bigcup_j L_j$. Now L_j is a singular 2-cell with boundary Bd $A_{4j+2} \cap$ Bd S_0. Since $S_0 \cup \bigcup_j L_j$ is tame, we may assume for the moment that this set is polyhedral relative to $K(M)$, and apply the Dehn Lemma of C. Papakyriakopoulos. (See, for example, [**MGT**, p. 199].) Thus every neighborhood of L_j contains a polyhedral 2-cell D_j such that

$$\text{Bd } D_j = \text{Bd } A_{4j+2} \cap \text{Bd } S_0, \qquad \text{Int } D_j \subset \text{Int } S_0.$$

Since $L_j \cap C_{4k} = \emptyset$, and $L_j \cap b = \emptyset$, it follows that D_j can be chosen for each j so that $D_j \cap C_{4k} = \emptyset$ for each k and $D_j \cap b = \emptyset$. But this is impossible: obviously $\bigcup_j D_j$ decomposes S_0 into 3-cells each of which contains exactly one set C_{4k}. (An elementary proof of this lemma is possible but tedious.)

LEMMA 9.1.5. *Every two different sets* Bd $S \cap C_{4r}$, Bd $S \cap C_{4s}$ *lie in different components of* $(S_1 - \text{Int } S_2) - \bigcup_j A'_{4j+2}$.

PROOF OF LEMMA. By Lemma 9.1.4, every connected set W in Bd S that intersects two different sets C_{4r}, C_{4s} must intersect L_j for some j. By Lemma 9.1.2, W intersects Int A'_{4j+2}. Therefore no component of Bd $S - \bigcup_j A'_{4j+2}$ intersects two different sets C_{4r}, C_{4s}. Since there are exactly m sets C_{4r}, and exactly m components of $(S_1 - \text{Int } S_2) - \bigcup_j A'_{4j+2}$, each of the former sets must lie in one of the latter; and the lemma follows.

Thus S has the following properties.

(1) $S \subset \text{Int } S_1$, $S_2 \subset \text{Int } S$.

(2) S and **B** satisfy the conditions for S_1 and **B** in Theorem 8.1.

(Since S has no cellular oscillations in **B**′, it follows that S has no cellular oscillations in **B**. Since $S \subset \text{Int } S_1$, S is "sufficiently small" so that S and **B**′ satisfy the other conditions for S_1 and **B** in Theorem 8.1. Therefore so also do S and **B**. But (2) as stated is what we need in Theorem 9.1. And the stronger condition using **B**′ would not necessarily be preserved by certain hypothetical operations presently to be discussed.)

(3) Different sets Bd $S \cap C'_{4k}$ lie in different components of $(S_1 - \text{Int } S_2) - \bigcup_j A'_{4j+2}$.

Evidently we may assume also that

(4) Bd S is in general position relative to each set A'_{4j+2}.

(If not, we move Pr Int A'_{4j+2} into general position relative to Pr Bd S, and lift.)

Finally we may suppose that S is chosen, subject to all the above conditions, so as to minimize the total number of components of all sets Bd $S \cap$ Int A'_{4j+2}. On this basis, we shall show that S is as in the conclusion of Theorem 9.1.

LEMMA 9.1.6. *Let J be a component of* Bd $S \cap$ Int A'_{4j+2}. *If J bounds a 2-cell in* Bd S, *then J bounds a 2-cell in A'_{4j+2}.*

PROOF OF LEMMA. Suppose not. Then J separates the two components of Bd A'_{4j+2} from one another in A'_{4j+2}. It follows immediately that the latitudinal polygon Bd $A'_{4j+2} \cap$ Bd S_1 is contractible in $S_1 - F$, which is absurd.

LEMMA 9.1.7. *Let J be a component of* Bd $S \cap$ Int A'_{4j+2}. *If J bounds a 2-cell in A'_{4j+2}, then J bounds a 2-cell in* Bd S.

PROOF OF LEMMA. Since Bd S contains a latitudinal polygon disjoint from J (Theorem 6.3), it follows that J is latitudinal in S or bounds a 2-cell in Bd S. The former is impossible, because J is contractible in $S - F$. The lemma follows.

LEMMA 9.1.8. *No component of* Bd $S \cap$ Int A'_{4j+2} *bounds a 2-cell in A'_{4j+2}.*

PROOF OF LEMMA. Suppose that $J =$ Bd D_J, $D_J \subset A'_{4j+2}$. Then J bounds a 2-cell $D'_J \subset$ Bd S, since otherwise a latitudinal polygon in Bd S would be contractible in $S_1 - F$. Now f is of period n, and both Bd S and A'_{4j+2} are f-invariant. We cannot have $f(D_J) = D_J$, since D_J contains no fixed point of f. And no set $f^r(D_J)$ can lie in the interior of another, since then the number of components of Bd $S \cap$ Int A'_{4j+2} would be infinite. Therefore the 2-cells $f^r(D_J)$ are disjoint, and are permuted by f. Thus Pr J is a component of Pr Bd $S \cap$ Pr A'_{4j+2}, and bounds 2-cells Pr D_J and Pr D'_J in Pr A'_{4j+2} and Pr Bd S respectively. We may suppose, without loss of generality, that J is inmost in A'_{4j+2}, in the sense that Int D_J contains no component of Bd $S \cap A'_{4j+2}$. It follows that Int Pr D_J contains no point of Pr Bd S. We substitute Pr D_J for Pr D'_J in Pr Bd S, force the resulting surface slightly off Pr A'_{4j+2} in the neighborhood of Pr D_J, and then move the resulting surface slightly (if need be), so as to restore general position relative to Pr Bd C'_{4j+2}. This gives a torus T. Since Pr Bd S and Pr D_J lie in Pr(Int $S_1 - S_2$), T can be chosen so as to lie in Pr(Int $S_1 - S_2$).

Let $\tilde{T} = \mathrm{Pr}^{-1}\, T$. Since Pr$|\tilde{T}$ is an n-sheeted covering of a torus, $\tilde{T}$ is a torus. Since T is polyhedral relative to $K(U)$, it follows that $\tilde{T}$ is polyhedral relative to $K(\tilde{U})$. Let S' be the component of $M - \tilde{T}$ that contains F. By Theorem 4.10, S' is a solid torus. Since

$$\mathrm{Bd}\ S' = \tilde{T} \subset \mathrm{Int}\ S_1 - S_2,$$

it follows that

(1) $S' \subset$ Int S_1, $S_2 \subset$ Int S.

We shall show that the other conditions for S are also preserved by the operation $S \to S'$.

Since $\mathbf{B}$ is a barrier system for S_1, and $S' \subset S_1$, we have $S' \subset \operatorname{Int} S_0$, and $A_{4k} \cap S' = \emptyset$ for each k. Also Bd S' is in general position relative to each set Bd C'_{4k}. Therefore $\mathbf{B}$ is a barrier system for S'. Since Bd $S' = \tilde{T} = \mathrm{Pr}^{-1}T$, it follows that Bd S' is f-invariant. Therefore so also is S'. We recall that $A'_{4j+2} \cap C_{4k} = \emptyset$ for each j and k. It follows that Bd $S' \cap C_{4k}$ is the union of some or all of the components of Bd $S \cap C_{4k}$. Since S has no cellular oscillations in $\mathbf{B}$, it follows that S' has the same property; the point is that if J is a polygon in Bd $S' \cap C'_{4k}$, and J bounds a 2-cell D_J in Bd S', then $J \subset$ Bd S, and J cannot be latitudinal in S, so that J bounds a 2-cell D'_J in Bd S. Since S has no cellular oscillations in $\mathbf{B}$, we have $D'_J \subset \bigcap_r f^r(C_{4k})$. Therefore $D'_J \subset$ Bd S', $D'_J = D_J$, and $D_J \subset \bigcup_r f^r(C_{4k})$. Since S_1 is "sufficiently small" in the sense of Theorem 8.1, and $S' \subset S_1$, it follows that S' is "sufficiently small." Thus we have:

(2) S' and $\mathbf{B}$ satisfy all the conditions for S_1 and $\mathbf{B}$ in Theorem 8.1.

It is easy to check, geometrically, that:

(3) Different sets Bd $S' \cap C'_{4k}$ lie in different components of $(S_1 - \operatorname{Int} S_2) - \bigcup_j A'_{4j+2}$.

To see this, regard the operation Pr Bd $S \to T$ as an operation Bd $S \to \tilde{T}$. Condition (2) is unaffected by (i) the deletion of the 2-cells $f^r(D_J)$, (ii) the addition of the 2-cells $f^r(D'_J)$, (iii) the "forcing off" process, and (iv) the restoration of general position.

Obviously the operation $S \to S'$ preserves (4); the effect is to delete certain components of Bd $S \cap A'_{4j+2}$.

But all this is impossible: it preserves (1)–(4), and reduces the total number of components of all sets Bd $S \cap A'_{4j+2}$. The lemma follows.

LEMMA 9.1.9. *Each set* Bd $S \cap A'_{4j+2}$ *has at most one component.*

PROOF OF LEMMA. We now know that every component of this set is latitudinal in S, and separates the components of Bd A'_{4j+2} from one another in A'_{4j+2}. Since every two of these components bound an annulus in A'_{4j+2}, and f is periodic, it follows that each of them is f-invariant. Suppose that there are more than one of them. Then their union decomposes Bd S into annuli with disjoint interiors, appearing in a certain cyclic order on Bd S. If each of these annuli approaches Int A'_{4j+2} locally from both sides, then it follows by an easy geometric argument that the number of these annuli is infinite, which is false. Therefore there is an annulus A_s in Bd S, with Bd $A_s \subset A'_{4j+2}$, such that A_s approaches Int A'_{4j+2} locally from only one side. Now Bd A_s bounds an annulus A'_s in Int A'_{4j+2}; and if two such annuli A'_s intersect, then one contains the other in its interior. Therefore we may choose A_s in such a way that $\text{Int } A'_s \cap \text{Bd } S = \emptyset$. Automatically we have

$$A'_s \cap (\text{Bd } S_1 \cup \text{Bd } S_2) = \emptyset.$$

We now simplify in much the same way as for 2-cells. Since each component of $\text{Bd } A_s = \text{Bd } A'_s$ is f-invariant, it follows that $\text{Pr } A_s$ and $\text{Pr } A'_s$ are annuli (with the same boundary). In $\text{Pr Bd } S$, we replace $\text{Pr } A_s$ by $\text{Pr } A'_s$, force the resulting surface off $\text{Pr Int } A'_{4j+2}$, and move the "new parts" of the new surface slightly, so as to get them into general position relative to $\text{Pr Bd } C'_{4j+2}$. Then we lift. As in the proof of the preceding lemma, this gives a torus $\tilde{T}$ which is the boundary of a solid torus S', and S' satisfies all the conditions for S; the verifications are substantially the same as in the preceding proof. As before, all this is impossible, because it reduces the total number of components of the sets $\text{Bd } S \cap A'_{4j+2}$.

LEMMA 9.1.10. *Each set* $\text{Bd } S \cap A'_{4j+2}$ *is a polygon.*

PROOF OF LEMMA. By (4) and the preceding lemma, it will be sufficient to show that $\text{Bd } S \cap A'_{4j+2} \neq \emptyset$. If this is false, then

$$\text{Bd } S \cap A'_{4j+2} = \text{Bd } S \cap L_j = \emptyset.$$

It follows, as in the proof of Lemma 9.1.4, that $\text{Bd } A_{4j+2} \cap \text{Bd } S_0$ bounds a tame 2-cell in $S_0 - S$, so that F lies in a 3-cell in S_0, and F is contractible in S_0, which is absurd.

Now the polygons $\text{Bd } S \cap \text{Int } A_{4j+2}$ decompose $\text{Bd } S$ into annuli N_j, with

$$\text{Bd } N_j = (A'_{4j-2} \cup A'_{4j+2}) \cap \text{Bd } S.$$

LEMMA 9.1.11. N_j *intersects* C_{4k} *only if* $k = j$.

PROOF OF LEMMA. We know that the sets L_j separate every two different sets C_{4k} from one another in S_0. By Theorem 7.1, for each j there is an f-invariant latitudinal polygon J_j in $\text{Bd } S \cap \text{Int } C_{4j}$. Let V_j be the component of $\text{Bd } S - \bigcup_r A'_{4r+2} = \text{Bd } S - \bigcup_r L_r$ that contains J_j. Then $\overline{V_j}$ is a 2-manifold with boundary, and $\text{Bd } \overline{V_j}$ consists of one or both of the polygons $\text{Bd } S \cap L_{j-1}$ and $\text{Bd } S \cap L_j$. If $\text{Bd } \overline{V_j}$ is connected, then the component of $S_0 - \bigcup_r L_r$ that contains C_{4j} intersects $\text{Bd } S$ in the union of two connected sets V_j, V'_j, and the closure of each of these is a 2-cell with handles. Since $\text{Bd } S$ is a torus, that is, a 2-cell with only one handle, it follows that at least one of the sets $\overline{V_j}$, $\overline{V'_j}$ is a 2-cell. It follows that a latitudinal polygon in $\text{Bd } S_0$ bounds a 2-cell in $S_0 - F$, which is impossible. Therefore $N_j = \overline{V_j}$ for each j, and the lemma follows.

Now for each j let J_j be an f-invariant latitudinal polygon in $\text{Bd } S \cap C_{4j}$, as in the proof of the preceding lemma. Then $J_j \subset \text{Int } N_j$ for each j, and J_j separates the two components of $\text{Bd } N_j$ from one another in N_j. Thus

$$N_j = N_j^- \cup N_j^+,$$

where N_j^- and N_j^+ are annuli, and

$$N_j^+ \cap N_j^- = J_j = \text{Bd } N_j^- \cap \text{Bd } N_j^+.$$

For each j, let

$$M_j = N_j^+ \cup N_{j+1}^-.$$

Then S and the sets M_j satisfy the conclusion of Theorem 9.1.

The notations $\mathbf{B}' = [\mathbf{P}, \mathbf{C}, \mathbf{C}', \mathbf{A}, S_0]$, $\mathbf{C} = \{C_{4j}\}$, $\mathbf{C}' = \{C'_{4j}\}$, and so on, in Theorem 9.1, were really part of the apparatus of the proof. The conclusion of the theorem mentions only $\mathbf{B}$, and not $\mathbf{B}'$. Moreover, given any barrier system $\mathbf{B}$ for F, there is a barrier system $\mathbf{B}'$ for F such that $\mathbf{B}$ and $\mathbf{B}'$ satisfy the conditions of Theorem 9.1. The theorem can therefore be restated more simply, as follows.

THEOREM 9.2. *Let* $\mathbf{B} = [\mathbf{P}, \mathbf{C}, \mathbf{C}', \mathbf{A}, S_0]$ *be a barrier system for* F, *with* $\mathbf{P} = \{P_j\}$, $\mathbf{C} = \{C_j\}$, $\mathbf{C}' = \{C'_j\}$, $\mathbf{A} = \{A_j\}$ $(1 \leqslant j \leqslant m)$. *Then every neighborhood of* F *contains a solid torus* S *such that*

(1) S *and* $\mathbf{B}$ *satisfy all the conditions for* S_1 *and* $\mathbf{B}$ *in Theorem* 8.1, *and*

(2) Bd S *is the union of a collection of annuli* M_j $(1 \leqslant j \leqslant m)$, *intersecting only on their boundaries, such that*

(2.1) M_j *is f-invariant,*

(2.2) Bd M_j *is the union of a latitudinal polygon in* $S \cap C_j$ *and a latitudinal polygon in* $S \cap C_{j+1}$, *and*

(2.3) M_j *intersects* C_k *only if* $k = j$ *or* $k = j + 1$.

THEOREM 9.3. *In Theorem* 9.2, $\delta M_j \leqslant 5\delta\mathbf{B}$.

Because M_j lies in the union of C_j, C_{j+1}, and the three components of $S_0 - \bigcup_r C_r$ that are contiguous to these two sets.

THEOREM 9.4. *Let* $\mathbf{B}$ *and* $\mathbf{B}'$ *be barrier systems for* F, *and let* S *be a solid torus such that* S *and* $\mathbf{B}'$ *satisfy all the conditions for* S *and* $\mathbf{B}$ *in Theorem* 9.2. *If the mesh* $\delta\mathbf{B}'$ *of* $\mathbf{B}'$ *is sufficiently small, then* S *has no cellular oscillations in* $\mathbf{B}$.

PROOF. Given $\mathbf{B} = [\mathbf{P}, \mathbf{C}, \mathbf{C}', \mathbf{A}, S_0]$, consider the pairs $C_j \in \mathbf{C}$, $C'_j \in \mathbf{C}'$. If $\delta\mathbf{B}'$ is sufficiently small, then

$$S \cap C'_j \subset \bigcap_r f^r(\text{Int } C_j).$$

If $\delta\mathbf{B}'$ is sufficiently small, then no annulus $M_j \subset$ Bd S (as in Theorem 9.2) intersects both a set C'_j and the corresponding set $M - \bigcap_r f^r(\text{Int } C_j)$.

If $\delta\mathbf{B}'$ is small in both these senses, then every polygon J in Bd $S \cap C'_j$ lies in an annulus A in Bd $S \cap \bigcap_r f^r(\text{Int } C_j)$, namely, the union A of all the annuli M_j that intersect J. If J bounds a 2-cell in S, then J bounds a 2-cell in A, and the theorem follows.

10. Transverse solid tori. If $\mathbf{B}$ and $\mathbf{B}'$ are related as in Theorem 9.1, then we shall say that $\mathbf{B}'$ is a *4-subdivision* of $\mathbf{B}$.

THEOREM 10.1. *There are sequences* $\mathbf{B}_1, \mathbf{B}_2, \ldots ; \mathbf{B}'_1, \mathbf{B}'_2, \ldots ; S_1, S_2, \ldots$ *such that the following conditons hold.*

(1) *For each* i, $\mathbf{B}_i$ *is a barrier system* $[\mathbf{P}_i, \mathbf{C}_i, \mathbf{C}'_i, \mathbf{A}_i, S_{0,i}]$ *for* F, *and* $\mathbf{B}'_i$ *is a 4-subdivision* $['\mathbf{P}_i, '\mathbf{C}_i, '\mathbf{C}'_i, '\mathbf{A}_i, S_{0,i}]$ *of* $\mathbf{B}_i$, *so that* $\mathbf{P}_i = \{P_{i,4j}\}$, $'\mathbf{P}_i = \{P_{ik}\}$ $(1 \leqslant j \leqslant m_i,\ 1 \leqslant k \leqslant 4m_i)$, *and similarly for* $\mathbf{C}_i$, $\mathbf{C}'_i$, *and so on.*

(2) *For each* i, $'\mathbf{P}_i \subset \mathbf{P}_{i+1}$.

(3) $\mathrm{Lim}_{i\to\infty}\, \delta_i = \mathrm{Lim}_{i\to\infty}\, \delta\mathbf{B}_i = 0$.

(4) *For each* i, S_i *and* $\mathbf{B}'_i$ *satisfy the conditions for* S *and* $\mathbf{B}$ *in Theorem* 9.2.

(5) *For each* i, $S_{i+1} \subset \mathrm{Int}\, S_i$.

(6) *For each* i *there is a collection* $\{A'_{ij}\}$ $(1 \leqslant j \leqslant m_i)$ *of disjoint annuli, such that*

(6.1) A'_{ij} *is* f-*invariant*,

(6.2) $A'_{ij} \subset S_i - \mathrm{Int}\, S_{i+1}$,

(6.3) $A'_{ij} \cap (\mathrm{Bd}\, S_i \cup \mathrm{Bd}\, S_{i+1}) = \mathrm{Bd}\, A'_{ij}$, *and this set is the union of a latitudinal polygon in* S_i *and a latitudinal polygon in* S_{i+1},

(6.4) $\mathrm{Bd}\, A'_{ij} \subset \mathrm{Int}\, C'_{4j}$,

(6.5) A'_{ij} *intersects* C_{ik} *only if* $k = 4j$ *or* $k = 4j \pm 1$, *and*

(6.6) A'_{ij} *is polyhedral relative to* $K(\tilde{U})$.

PROOF. Obviously, by repeated applications of Theorem 6.1, we can get sequences $\mathbf{B}_1, \mathbf{B}_2, \ldots$ and $\mathbf{B}'_1, \mathbf{B}'_2, \ldots$ satisfying (1), (2), and (3); and we can make the diameters $\delta_i = \delta\mathbf{B}_i$ as small as we please.

We define the three desired sequences recursively, as follows.

(i) Define $\mathbf{B}_1$, $\mathbf{B}'_1$, with $\delta\mathbf{B}_1 < 1$. By Theorem 9.2 there is an S_1 such that S_1 and $\mathbf{B}'_1$ satisfy the conditions for S and $\mathbf{B}$ in Theorem 9.2. We take S_1 in a sufficiently small neighborhood of F so that S_1 and $\mathbf{B}'_1$ satisfy the conditions for S_1 and $\mathbf{B}$ in Theorem 8.1.

(ii) Suppose that we have given $\mathbf{B}_i$, $\mathbf{B}'_i$, and S_i for $i \leqslant i_0$, and $\{A'_{ij}\}$ for $i < i_0$, satisfying (1)–(6). Define $\mathbf{B}_{i_0+1}$, B'_{i_0+1} so as to preserve (1) and (2), making $\delta_{i_0+1} = \delta\mathbf{B}_{i_0+1}$ sufficiently small so that if S' and B'_{i_0+1} satisfy the conditions for S and $\mathbf{B}$ in Theorem 9.2, then S' has no cellular oscillations in $\mathbf{B}'_{i_0}$. (By Theorem 9.4, this can be done.) By the induction hypothesis, S_{i_0} and $\mathbf{B}'_{i_0}$ satisfy the conditions for S_1 and $\mathbf{B}$ in Theorem 8.1. Let S_{i_0+1} be a solid torus such that S_{i_0+1} and $\mathbf{B}'_{i_0+1}$ satisfy the conditions for S and $\mathbf{B}$ in Theorem 9.2. Then S_{i_0}, S_{i_0+1}, and $\mathbf{B}'_{i_0}$ satisfy the conditions for S_1, S_2, and $\mathbf{B}$ in Theorem 8.1. Applying Theorem 8.1, we get a collection $\{A'_j\}$ of annuli. Of these, we use only the annuli $A'_{4j} = A'_{i_0 j}$. By (5) of Theorem 8.1, different sets $A'_{i_0 j}$ are disjoint. (6.1)–(6.6) now follow directly from (1)–(6) of Theorem 8.1.

Hereafter, the sequences described in Theorem 10.1 will be regarded as fixed.

THEOREM 10.2. *For each i, j,*

$$A'_{ij} \subset N(P_{i,4j}, 3\delta_i).$$

This is true because A'_{ij} lies in the union of $C_{i,4j}$ and the two components of $S_{0,i} - \bigcup_r C_{i,4r}$ that are contiguous to $C_{i,4j}$.

THEOREM 10.3. *For each i, $\mathrm{Bd}\, S_i$ is the union of a collection of annuli M_{ij} ($1 \leqslant j \leqslant m_i$), intersecting only on their boundaries, such that* (1) *M_{ij} is f-invariant,* (2) *$\mathrm{Bd}\, M_{ij}$ is the union of a latitudinal polygon in $S_i \cap \mathrm{Int}\, C_{i,4j}$ and a latitudinal polygon in $S_i \cap \mathrm{Int}\, C_{i,4j+4}$, and* (3) *$M_{ij}$ intersects C_{ik} only for $4j \leqslant k \leqslant 4j + 4$. Moreover, $\mathrm{Bd}\, S_i$ is the union of a collection $\{N_{ij}\}$ of annuli, intersecting only on their boundaries, such that* (4) *for each i, j, $\mathrm{Bd}\, S_i \cap C_{i,4j} \subset \mathrm{Int}\, N_{ij}$ and* (5) *N_{ij} intersects C_{ik} only for $4j - 2 \leqslant k \leqslant 4j + 2$.*

PROOF. Let $\{M'_j\}$ be a collection of annuli ($1 \leqslant j \leqslant 4m_i$) such that M'_j, S_i, and $\mathbf{B}'_i$ satisfy the conditions of Theorem 9.2. Let

$$M_{ij} = \bigcup_{i=4j}^{4j+3} M'_j, \qquad N_{ij} = \bigcup_{r=4j-2}^{4j+1} M'_j.$$

For each i, j, let $J_{ij} = M_{i,j-1} \cap M_{ij}$. For each i, j, there is exactly one $k = k(i, j)$ such that $P_{i+1,k} = P_{i,4j}$.

THEOREM 10.4. *For each i, j ($1 \leqslant j \leqslant m_i$), $J_{ij} \cup J_{i+1,k}$ is the boundary of an annulus A''_{ij} such that*

(1) *A''_{ij} is a polyhedron relative to $K(\tilde{U})$,*

(2) *A''_{ij} is f-invariant,*

(3) $\mathrm{Int}\, A''_{ij} \subset \mathrm{Int}\, S_i - S_{i+1}$,

(4) *A''_{ij} intersects $C_{i,4r}$ only if $r = j$, and*

(5) $A''_{ij} \subset N(P_{i,4j}, 4\delta_i)$.

Moreover, the annuli A''_{ij} can be chosen so as to be disjoint.

PROOF. We start with the annuli A'_{ij} of Theorem 10.1, so that

$$\mathrm{Bd}\, A'_{ij} \subset \mathrm{Int}\, C'_{i,4j}.$$

It follows that

$$J'_{ij} = A'_{ij} \cap \mathrm{Bd}\, S_i \subset \mathrm{Int}\, N_{ij}, \qquad N_{ij} \cap A'_{ik} = \varnothing \quad \text{for } k \neq j.$$

By Theorem 10.3, we have $J_{ij} \subset \mathrm{Int}\, N_{ij}$. Therefore $\mathrm{Pr}\, J_{ij}$, $\mathrm{Pr}\, J'_{ij} \subset \mathrm{Pr}\, \mathrm{Int}\, N_{ij}$. Since $\mathrm{Pr}\, N_{ij}$ is an annulus, it follows by Theorem 4.11 that there is a PL homeomorphism h_{ij}: $U \leftrightarrow U$, $\mathrm{Pr}\, N_{ij} \leftrightarrow \mathrm{Pr}\, N_{ij}$, such that $h_{ij}|\mathrm{Pr}\, \mathrm{Bd}\, N_{ij}$ is the identity and $h_{ij}(\mathrm{Pr}\, J'_{ij}) = \mathrm{Pr}\, J_{ij}$; and h_{ij} can be chosen so as to differ from the identity only in an arbitrarily small neighborhood of $\mathrm{Pr}\, \mathrm{Int}\, N_{ij}$. Thus we may assume that $h_{ij}|\mathrm{Pr}\, S_{i+1}$ and $h_{ij}|\mathrm{Pr}(\mathrm{Bd}\, S_i - N_{ij})$ are identity mappings. Let

$$A = h_{ij}(A'_{i,4j}), \qquad \tilde{A} = \mathrm{Pr}^{-1}A.$$

Let W_{ij} be the set on which h_{ij} differs from the identity. Since N_{ij} lies in the union V_{ij} of $C_{i,4j}$ and the two adjacent components of $S_i - \bigcup_r C_{i,4r}$, it follows that W_{ij} can be chosen so that

$$\tilde{W}_{ij} = \mathrm{Pr}^{-1}W_{ij} \subset V_{ij}.$$

It follows that $\tilde{A}$ intersects C_{ir} only if $4j - 2 \leqslant r \leqslant 4j + 2$.

Now consider the other component $A'_{ij} \cap \mathrm{Bd}\, S_{i+1}$ of $\mathrm{Bd}\, A'_{ij}$. This is also a component of $\mathrm{Bd}\, \tilde{A}$. Let $J'_{i+1,k} = S_{i+1} \cap \mathrm{Bd}\, \tilde{A}$. Let r_0 and r_1 be the integers such that $P_{i,4j-2} = P_{i+1,r_0}$ and $P_{i,4j+2} = P_{i+1,r_1}$. Let

$$N = \bigcup_{r=r_0}^{r_1} M_{i+1,r}.$$

Then N is an annulus, and contains $C_{i,4j} \cap \mathrm{Bd}\, S_{i+1}$ in its interior. Thus

$$J'_{i+1,k}, J_{ik} \subset \mathrm{Int}\, N, \qquad N \cap \mathrm{Bd}\, A'_{ik} = \varnothing \quad \text{for } k \neq j,$$

and

$$\mathrm{Pr}\, J'_{i+1,k}, \quad \mathrm{Pr}\, J_{ik} \subset \mathrm{Pr}\, \mathrm{Int}\, N.$$

As before, we move $\mathrm{Pr}\, J'_{i+1,k}$ onto $\mathrm{Pr}\, J_{ik}$ by a PL homeomorphism h'_{ij}: $U \leftrightarrow U$, $\mathrm{Pr}\, \mathrm{Int}\, N \leftrightarrow \mathrm{Pr}\, \mathrm{Int}\, N$, such that $h'_{ij}|\mathrm{Pr}(\mathrm{Bd}\, S_{i+1} - N)$ is the identity and such that h'_{ij} differs from the identity only in a small neighborhood of $\mathrm{Pr}\, \mathrm{Int}\, N$. Since N intersects C_{ir} only if $r_0 \leqslant r \leqslant r_1$, and $\tilde{A}$ has the same property, it follows that h'_{ij} can be chosen so that the set

$$A''_{ij} = \mathrm{Pr}^{-1}h'_{ij}(A)$$

intersects $C_{i,4r}$ only if $r = j$. It follows that A''_{ij} lies in the union W of $C_{i,4j}$ and the components of $S_i - \bigcup_r C_{i,4r}$ that are contiguous to $C_{i,4j}$. Therefore A''_{ij} lies in the closure of $N(P_{i,4j}, 3\delta_i)$, and hence in $N(P_{i,4j}, 4\delta_i)$. And if our homeomorphisms are chosen so that the sets on which they differ from the identity are disjoint, then for each i, different sets A''_{ij} will be disjoint. This completes the proof of Theorem 10.4.

Now the union of the annuli A''_{ij} decomposes the toroidal shell $S_i - \mathrm{Int}\, S_{i+1}$ into m_i 3-manifolds S_{ij} with boundary, each of these being polyhedral relative to $K(\tilde{U})$. Each S_{ij} is bounded by the union of (a) the annulus M_{ij} in $\mathrm{Bd}\, S_i$, (b) an annulus in $\mathrm{Bd}\, S_{i+1}$, and (c) the union of two successive annuli $A''_{ij}, A''_{i,j+1}$.

THEOREM 10.5. *Each S_{ij} is f-invariant.*

Because $\mathrm{Bd}\, S_{ij}$ is f-invariant.

THEOREM 10.6. $S_1 = \bigcup_{i,j} S_{ij} \cup F$.

PROOF. Evidently

$$S_1 = \bigcup_i [S_i - \text{Int } S_{i+1}] \cup F.$$

Since for each i the sets S_{ij} form a decomposition of $S_i - \text{Int } S_{i+1}$, the theorem follows.

THEOREM 10.7. $\text{Pr } S_1 = \bigcup_{i,j} \text{Pr } S_{ij} \cup \text{Pr } F$.

Proof by Theorem 10.6.

THEOREM 10.8. *For each* i, j, $S_{ij} \subset N(P_{i,4j}, 5\delta_i)$.

PROOF. Let r_0 and r_1 be the integers such that $P_{i,4j} = P_{i+1,r_0}$, $P_{i,4j+4} = P_{i+1,r_1}$. Let

$$M'_{ij} = \bigcup_{r=r_0}^{r_1-1} M_{i+1,r},$$

so that

$$\text{Bd } S_{ij} = M_{ij} \cup A''_{ij} \cup A''_{i,j+1} \cup M'_{ij}.$$

Now M_{ij} intersects $C_{i,4r}$ only for $r = j, j+1$; A''_{ij} intersects $C_{i,4r}$ only for $r = j$; and $A''_{i,j+1}$ intersects $C_{i,4r}$ only for $r = j+1$. Applying Theorem 10.3 to the sets $M_{i+1,r} \subset M'_{ij}$ and the collection $\mathbf{C}_{i+1}$, it is also easy to check that $M_{i+1,r}$ intersects $C_{i,4s}$ only for $s = j, j+1$. Thus $\text{Bd } S_{ij}$ lies in the union W of $C_{i,4j}$, $C_{i,4j+4}$, and the components of $S_i - \bigcup_r C_{i,4r}$ that are contiguous to $C_{i,4j}$ and $C_{i,4j+4}$. Now $\overline{W}$ is a 3-cell; and since $\text{Bd } S_{ij} \subset \overline{W}$, it follows that $S_{ij} \subset \overline{W}$. An easy computation gives $\overline{W} \subset N(P_{i,4j}, 5\delta_i)$, and the theorem follows.

THEOREM 10.9. *Each* S_{ij} *is a solid torus.*

PROOF. We know (by the hypothesis of Theorem 1.1) that F is the boundary of a tame 2-cell $D \subset \mathbf{S}^3$. Let $K(\mathbf{S}^3)$ be a triangulation of $\mathbf{S}^3$. Then there is a homeomorphism $h\colon \mathbf{S}^3 \leftrightarrow \mathbf{S}^3$, mapping D onto a set which is polyhedral relative to $K(\mathbf{S}^3)$; and obviously h can be chosen so that $h(D) \subset \tilde{U}$. (If not, shrink $h(D)$ onto a set of small diameter, and move the resulting set into $\tilde{U}$.) Let $K_1(\tilde{U})$ be a triangulation of $\tilde{U}$ in which each simplex is a linear subsimplex of a simplex of $K(\mathbf{S}^3)$. Then $D \cap \tilde{U}$ is semilocally tame in $\tilde{U}$ relative to $K_1(\tilde{U})$. By Theorem 3.4, $D \cap \tilde{U}$ is tame in $\tilde{U}$ relative to $K_1(\tilde{U})$. By the Hauptvermutung, $D \cap \tilde{U}$ is tame in $\tilde{U}$ relative to $K(\tilde{U})$. By Theorem 3.4, there is a homeomorphism $g\colon \tilde{U} \leftrightarrow \tilde{U}$, such that $g(D \cap \tilde{U})$ is polyhedral relative to $K(\tilde{U})$; and g may be chosen so that there is an extension $g'\colon \mathbf{S}^3 \leftrightarrow \mathbf{S}^3$ of g, such that $g'|F$ and $g'|(\mathbf{S}^3 - M)$ are identity mappings. Therefore we may assume that the D that was given has the property that $D \cap \tilde{U}$ is polyhedral relative to $K(\tilde{U})$; and of course we may suppose that Int D is in general position relative to Bd S_{ij}. Subject to all these conditions, we choose D so as to minimize the number q of components of $D \cap$ Bd S_{ij}. Each of these components is of course a polygon.

LEMMA 10.9.1. $q > 0$.

PROOF OF LEMMA. If $q = 0$, then S_{ij} lies in a 3-cell in $\mathbf{S}^3 - F$, and so Bd $A''_{ij} \cap$ Bd S_i is contractible in $\mathbf{S}^3 - F$. Since this set is a latitudinal polygon in S_i, this contradicts Theorem 4.9.

LEMMA 10.9.2. *No polygon J in $D \cap$ Bd S_{ij} bounds a 2-cell in* Bd S_{ij}.

PROOF OF LEMMA. Suppose that $J =$ Bd D_J, where D_J is a 2-cell in Bd S_{ij}; and suppose that J is inmost in Bd S_{ij}, in the sense that Int D_J contains no other component of $D \cap$ Bd S_{ij}. Then D can be "forced off Bd S_{ij} in the neighborhood of D_J." This is impossible, because it reduces q.

Now let J be a component of $D \cap$ Bd S_{ij} which is inmost in D, so that J bounds a 2-cell $D_J \subset$ Int D such that Int $D_J \cap$ Bd $S_{ij} = \varnothing$. If Int $D_J \subset \mathbf{S}^3 - S_{ij}$, then S_{ij} lies in a 3-cell in $\mathbf{S}^3 - F$, which is impossible, as in the proof of Lemma 10.9.1. Therefore Int $D_J \subset$ Int S_{ij}. When S_{ij} is split apart at D_J, the result is a 3-cell. Reidentifying, we find that S_{ij} is either a solid torus (with J a latitudinal polygon in S_{ij}) or a solid Klein Bottle. The latter is impossible, because S_{ij} is a subspace of $\mathbf{S}^3$; and Theorem 10.9 follows.

THEOREM 10.10. *Each set* Pr S_{ij} *is a solid torus.*

PROOF. Evidently Pr S_{ij} is a 3-manifold with boundary, polyhedral relative to $K(U)$, and Bd Pr S_{ij} = Pr Bd S_{ij} is a torus. Since S_{ij} is a solid torus, there is a loop L in Bd Pr S_{ij} which is contractible in Pr S_{ij} but not in Bd Pr S_{ij}. By the Loop Theorem it follows that there is a PL 2-cell Δ_{ij} in Pr S_{ij} such that

$$\text{Bd } \Delta_{ij} = \Delta_{ij} \cap \text{Bd Pr } S_{ij},$$

and such that Bd Δ_{ij} is not contractible in Bd Pr S_{ij}. Consider the set $\text{Pr}^{-1}\Delta_{ij}$. Since Int Δ_{ij} and Bd Δ_{ij} are locally Euclidean, and Pr is a local homeomorphism, it follows that (a) Pr^{-1}Bd Δ_{ij} is the union of n disjoint polygons and (b) Pr^{-1}Int Δ_{ij} is the union of n disjoint sets homeomorphic to Int Δ_{ij}. Thus (c) $\text{Pr}^{-1}\Delta_{ij} = \bigcup_{r=1}^{n} d_r$, where the sets d_r are disjoint 2-cells, with

$$d_r \cap \text{Bd } S_{ij} = \text{Bd } d_r, \qquad \text{Int } d_r \subset \text{Int } S_{ij},$$

so that the polygons Bd d_r are latitudinal in S_{ij}. Thus the sets d_r decompose S_{ij} into n 3-cells, each of which is mapped onto Pr S_{ij} under a mapping which identifies two disjoint 2-cells in its boundary. It follows, as in the proof of the preceding theorem, that Pr S_{ij} is either a solid torus or a solid Klein Bottle. Thus, to complete the proof of Theorem 10.10, it will be sufficient to show that Pr S_{ij} is orientable.

Now Bd $S_{ij} \cap$ Bd S_{i+1} is tame relative to $K(\tilde{U})$, and hence tame relative to a triangulation $K(\mathbf{S}^3)$ of $\mathbf{S}^3$; and every polygon J in the annulus $A = S_{ij} \cap$ Bd S_{i+1} either bounds a 2-cell in A or is latitudinal in S_{i+1}. It follows, by a straightforward construction, that there is a tame 3-cell C^3 in $\mathbf{S}^3$ such that $C^3 \cap C_{ij} = A$, and such that Bd $C^3 \cup$ Bd S_{ij} is also tame. Thus the set $C_1^3 = S_{ij} \cup C^3$ is a 3-cell. Let g: $C_1^3 \leftrightarrow C_1^3$ be an extension of $f|S_{ij}$, such that g is tame, and such that $g|C^3$ is conjugate to a rotation of a cylinder $\mathbf{B}^2 \times [0, 1]$ about its axis of symmetry. Thus the fixed-point set of g is a broken line B, with its two end-points in the two components of Bd $C_1^3 - A$; and B is unknotted in C_1^3, in the obvious sense. Now let C_2^3 be a combinatorial copy of C_1^3, with identifications such that Bd C_2^3 = Bd C_1^3 and $C_1^3 \cup C_2^3$ is a 3-sphere S^3. Now extend g so as to get a homeomorphism g': $S^3 \leftrightarrow S^3$, simplicial relative to some triangulation of S^3, and with an unknotted polygon as its fixed-point set. Any such g' is periodic, with the same period as f. The main result of [M] asserts that under these conditions g' is conjugate to a rotation. It follows that the orbit space Ω'' of g' is a 3-sphere. Since Pr S_{ij} is homeomorphic to a subspace of Ω'', it follows that Pr S_{ij} is orientable; and this is sufficient to complete the proof of Theorem 10.10.

THEOREM 10.11. *In each set* Pr S_{ij} *there is a* PL *2-cell* Δ_{ij} *such that*

(1) Bd $\Delta_{ij} = \Delta_{ij} \cap$ Bd Pr S_{ij},

(2) Bd Δ_{ij} *is latitudinal in* Pr S_{ij},

(3) Bd Δ_{ij} *is in general position relative to each set* Pr J_{rs} *that lies in* Bd Pr S_{ij}, *and*

(4) $\mathrm{Pr}^{-1}\, \Delta_{ij}$ *is the union of n disjoint 2-cells* d_r, *with* Bd $d_r = d_r \cap$ Bd S_{ij}.

PROOF. In the proof of Theorem 10.10 we got 2-cells Δ_{ij} satisfying (1) and (4), such that Bd Δ_{ij} is not contractible in Bd Pr S_{ij}. Therefore each Δ_{ij} also satisfies (2). These properties are preserved by any PL homeomorphism Pr $S_{ij} \leftrightarrow$ Pr S_{ij}. By such a homeomorphism, we move each set Bd Δ_{ij} slightly so as to get condition (3).

In the following definition, the sets A''_{ij} are as in Theorem 10.4.

Given A''_{ij}, let $j_0 = j$; and for each $k \geqslant 0$ let j_k be the integer such that $P_{i,4j_0} = P_{i+k,4j_k}$. (There is such a j_k; see (2) of Theorem 10.1.) We form the sequence

$$A''_{i,j_0}, A''_{i+1,j_1}, \ldots;$$

and let

$$D_{ij} = \bigcup_{k=0}^{\infty} A''_{i+k,j_k} \cup \{P_{i,4j_0}\}.$$

Under our conditions for the sets A''_{ij}, we have

$$A''_{i,j_k} \cap A''_{i,j_{k+1}} = J_{i,j_{k+1}}.$$

By Theorem 10.7 we have

$$A''_{i+k,j_k} \subset N(P_{i+k,4j_k}, 4\delta_{i+k}) = N(P_{i,4j_0}, 4\delta_{i+k}).$$

Therefore

$$\bigcup_{k=r}^{\infty} A''_{i+k,j_k} \subset N(P_{i,4j_0}, 4\delta_{i+r})$$

for each r. Therefore

$$\mathrm{Cl}\left[\bigcup_{k=0}^{\infty} A''_{i+k,j_k}\right] = \bigcup_{k=0}^{\infty} A''_{i+k,j_k} \cup \{P_{i,4j_0}\};$$

and it follows by a straightforward construction that D_{ij} is a 2-cell. We recall that

$$J_{ij} = M_{i,j-1} \cap M_{ij} = \mathrm{Bd}\, A''_{ij} \cap \mathrm{Bd}\, S_i.$$

Thus

$$\mathrm{Bd}\, D_{ij} = J_{ij}$$

for each i, j.

THEOREM 10.12. *For each i, j, J_{ij} is longitudinal in S_{ij}.*

PROOF. We have $J_{ij} = \mathrm{Bd}\, D_{ij}$, $D_{ij} \cap S_{ij} = A''_{ij} \subset \mathrm{Bd}\, S_{ij}$. It follows immediately that J_{ij} is contractible in $M - \mathrm{Int}\, S_{ij}$. Now apply Theorem 4.9.

THEOREM 10.13. *The 2-cells Δ_{ij} of Theorem 10.11 can be chosen in such a way that each intersection $\mathrm{Bd}\, \Delta_{ij} \cap \mathrm{Pr}\, J_{ij}$ is a single point.*

PROOF. Let $\{d_r\}$ be as in Theorem 10.11. Suppose that the sets Δ_{ij} are chosen (subject to the conditions of Theorem 10.11) so as to minimize the number of points in $\mathrm{Bd}\, \Delta_{ij} \cap \mathrm{Pr}\, J_{ij}$. On this basis we shall show that the theorem follows.

Now $\mathrm{Bd}\, d_r$ is latitudinal in S_{ij}, J_{ij} is longitudinal in S_{ij}, and these polygons are in general position relative to one another in $\mathrm{Bd}\, S_{ij}$. By Theorem 4.6 it follows that these polygons cross each other algebraically once.

Suppose now that the theorem is false, so that $\mathrm{Bd}\, \Delta_{ij} \cap \mathrm{Pr}\, J_{ij}$ contains more than one point. It follows that $\mathrm{Bd}\, d_r$ intersects J_{ij} in more than one point. From this we can easily show that there is a broken line $b_r \subset \mathrm{Bd}\, d_r$, and a broken line $b_{ij} \subset J_{ij}$, such that $b_r \cup b_{ij}$ bounds a 2-cell d in $\mathrm{Bd}\, S_{ij}$, with $\mathrm{Bd}\, d \cap J_{ij} = b_{ij}$. Since different sets d_r are disjoint, J_{ij} is f-invariant, and f is periodic, it follows that the images $f^r(d)$ are disjoint. Therefore $\mathrm{Pr}|d$ is a homeomorphism. Therefore $\mathrm{Pr}\, b_r$ can be dragged across $\mathrm{Pr}\, J_{ij}$, in the neighborhood of $\mathrm{Pr}\, d$, by a PL homeomorphism $\mathrm{Pr}\, S_{ij} \leftrightarrow \mathrm{Pr}\, S_{ij}$, $\mathrm{Bd}\, \mathrm{Pr}\, S_{ij} \leftrightarrow \mathrm{Bd}\, \mathrm{Pr}\, S_{ij}$ which differs from the identity only in a small neighborhood of d. All this is impossible, because it reduces the number of components of $\mathrm{Bd}\, \Delta_{ij} \cap \mathrm{Pr}\, J_{ij}$.

THEOREM 10.14. *For each* i, j, $\operatorname{Pr} J_{ij}$ *is longitudinal in* $\operatorname{Pr} S_{ij}$.

PROOF. We know that $\operatorname{Bd} \Delta_{ij}$ is latitudinal in S_{ij}, and that J_{ij} crosses $\operatorname{Bd} \Delta_{ij}$ in exactly one point. When we split $\operatorname{Pr} S_{ij}$ apart at Δ_{ij}, $\operatorname{Pr} J_{ij}$ appears as a broken line joining points of the two 2-cells which have the same image when we reidentify. It follows that $\operatorname{Pr} J_{ij}$ carries a generator of $H_1(\operatorname{Pr} S_{ij})$.

THEOREM 10.15. *Every polygon* $\operatorname{Pr} J_{rs}$ *that lies in* $\operatorname{Bd} \operatorname{Pr} S_{ij}$ *is longitudinal in* $\operatorname{Pr} S_{ij}$.

PROOF. There are finitely many such polygons; they are disjoint; none are contractible in $\operatorname{Bd} \operatorname{Pr} S_{ij}$; and one of them ($\operatorname{Pr} J_{ij}$) is longitudinal. Therefore so also are the others. (Theorem 4.3.)

11. A triangulation of Ω. For each i, j, we define a sequence $S_{i,j,0}, S_{i,j,1}, \ldots,$ inductively, as follows. (1) Let $S_{i,j,0} = S_{ij}$. (2) Given a solid torus $S_{ijk} \subset S_{i+k} - \operatorname{Int} S_{i+k+1}$, intersecting $\operatorname{Bd} S_{i+k+1}$ in an annulus which forms a union of annuli $M_{i+k+1,r}$, let $S_{i,j,k+1}$ be the union of all solid tori $S_{i+k+1,r}$ which have an annulus in common with S_{ijk}.

For each i, j, let B_{ij} be the "short arc" in F between $P_{i,4j}$ and $P_{i,4j+4}$ that is, the arc which contains no third point $P_{i,4r}$. Let

$$D_{ij} = \bigcup_{k=0}^{\infty} S_{ijk} \cup B_{ij}.$$

THEOREM 11.1. *Each set* D_{ij} *is compact.*

PROOF. By induction on i, it is easy to check that if $S_{i+k,r} \subset S_{ijk}$, then $P_{i+k,4r} \in B_{ij}$. By Theorem 10.8 it follows that

$$S_{i+k,r} \subset N(P_{i+k,4r}, 5\delta_{i+k}) \subset N(B_{ij}, 5\delta_{i+k}),$$

where $N(X, \varepsilon)$ is the ε-neighborhood of the set X. Since the sequence $\delta_1, \delta_2, \ldots$ is decreasing, it follows that

(a) $\bigcup_{k=k_0}^{\infty} S_{ijk} \subset N(B_{ij}, 5\delta_{i+k})$ for each k_0. Since $\bigcup_{k=0}^{k_0} S_{ijk}$ is closed for each k_0, it follows that

$$\operatorname{Cl}\left[\bigcup_{k=0}^{\infty} S_{ijk}\right] = \bigcup_{k=0}^{\infty} S_{ijk} \cup B_{ij},$$

and the theorem follows.

THEOREM 11.2. *For each* i, j, $\delta B_{ij} \leqslant 3\delta_i$.

(Because B_{ij} lies in the union of $C_{i,4j}$, $C_{i,4j+4}$, and the component of $S_{0,i} - \bigcup_r C_{i,4r}$ that lies between them, and each of these sets has diameter no greater than δ_i.)

THEOREM 11.3. $\text{Lim}_{i\to\infty} \delta D_{ij} = 0$, *uniformly with respect to j. That is, for every* $\varepsilon > 0$ *there is an* i_0 *such that if* $i \geqslant i_0$, *then* $\delta D_{ij} < \varepsilon$ *for every j.*

PROOF. Setting $k = 0$ in (a) above, we get $D_{ij} \subset N(B_{ij}, 5\delta_i)$. If i is sufficiently large, then $\delta B_{ij} < \varepsilon/3$, $5\delta_i < \varepsilon/3$. It follows that $\delta D_{ij} < \varepsilon$, as desired.

For each i, j, let $D'_{ij} = \text{Pr } D_{ij}$.

THEOREM 11.4. $\text{Lim}_{i\to\infty} \delta D'_{ij} = 0$, *uniformly with respect to j.*

This is true because Pr is uniformly continuous.

THEOREM 11.5. Pr S_1 *is a solid torus.*

PROOF. Let F' be the circle in the xz-plane in $\mathbf{R}^3$, with center at (0, 0, 0) and radius 2. For each i, let D_i be the closed circular region in the xy-plane, with center at (2, 0, 0) and radius $1/i$; and let S'_i be the solid of revolution obtained by rotating D_i about the y-axis. Thus S'_i is a solid torus with F' as a spine, and $\bigcap_i S'_i = F'$.

Let $\mathbf{P}'_1$ be a set of m_1 points $P'_{1,4j}$ of F', equally spaced on F'. (We recall that m_i is the number of points in $\mathbf{P}_i$.) Given an ascending sequence $\mathbf{P}'_1, \mathbf{P}'_2, \ldots, \mathbf{P}'_i$, such that $\bigcup_{k=1}^{i} \mathbf{P}'_k$ is the image of $\bigcup_{k=1}^{i} \mathbf{P}_k$ under a bijection $\mathbf{P}_k \leftrightarrow \mathbf{P}'_k$ which preserves cyclic order on F and F', we form $\mathbf{P}'_{i+1}$ so as to preserve this property, in such a way that the points of $\mathbf{P}'_{i+1}$ that lie on each "short arc" from $P'_{i,4j}$ to $P'_{i,4j+4}$ are equally spaced on the short arc. Thus the bijection $\mathbf{P}_i \leftrightarrow \mathbf{P}'_i$ can be extended to give a bijection $\mathbf{P}_{i+1} \leftrightarrow \mathbf{P}'_{i+1}$, preserving cyclic order. By (2) of Theorem 10.1, $\mathbf{P}'_{i+1}$ decomposes F' into arcs which are at most one-fourth as long as the longest short arc of the type $P_{i,4j}\, P_{i,4j+4}$. It follows that $\bigcup_i \mathbf{P}'_i$ is dense in F'.

For each i, j, let H_{ij} be the half-plane with the y-axis as edge, passing through $P'_{i,4j}$. Then for each i, $\{H_{ij}\}$ decomposes Bd S'_i into annuli M'_{ij}; H_{ij} intersects $S'_i - \text{Int } S'_{i+1}$ in an annulus A'''_{ij}; and the sets A'''_{ij} decompose $S'_i - \text{Int } S'_{i+1}$ into solid tori S'_{ij}.

It is a straightforward matter to define a triangulation $K(S'_1)$ of S'_1 (obviously not rectilinear in $\mathbf{R}^3$) such that (1) F' forms a subcomplex of $K(S'_1)$ and (2) each set $M'_{1,j}$ forms a subcomplex of $K(S'_1)$. Evidently $K(S'_1)$ gives a PL structure in S'_1.

Evidently there is a homeomorphism

$$h_1\colon \text{Bd } S'_1 \leftrightarrow \text{Bd Pr } S_1,$$

such that

$$h_1(M'_{1,j}) = \text{Pr } M_{1,j} \qquad (1 \leqslant j \leqslant m_1);$$

and h_1 can be constructed so as to be PL relative to $K(S'_1)$ and $K(U)$.

Construction. For each i, j, let

$$J'_{ij} = M'_{i,j-1} \cap M'_{ij} = \text{Bd } A'''_{ij} \cap \text{Bd } S'_i.$$

First define h_1 on $\bigcup_j J'_{1,j}$, so that $h_1(J'_{1,j}) = \text{Pr } J_{1,j}$ for each j. Then, in each set $M'_{1,j}$, extend h_1 to two disjoint broken lines in $M'_{1,j}$, joining the two components of Bd $M'_{1,j}$, and intersecting Bd $M'_{1,j}$ only in their end-points; these are mapped by the extended h_1 onto broken lines in the corresponding sets Pr $M_{1,j}$. It is easy to ensure that the extended h_1 is PL. Now each of the sets $M'_{1,j}$, Pr $M_{1,j}$ is decomposed into two combinatorial 2-cells; and for appropriate choice of the initial h_1 ("preserving orientation"), the boundaries of the two 2-cells in $M'_{1,j}$ will be mapped onto the boundaries of the two 2-cells in Pr $M_{1,j}$. Therefore h_1 can be extended again so as to give the desired PL homeomorphism h_1: Bd $S'_1 \leftrightarrow$ Bd Pr S_1.

In further extensions of h_1, we shall not care whether our mappings are PL.

The xz-plane intersects each set S'_{ij} in the union of two 2-cells $d'_{i,j,1}$, $d'_{i,j,2}$, with $d'_{i,j,1}$ lying inside F' and $d'_{i,j,2}$ lying outside F'. (The geometry here is clear, and so we need not list the ways in which these 2-cells intersect the sets already defined in S'_1.) We have already defined

$$h_1\colon \text{Bd } S'_1 \leftrightarrow \text{Bd Pr } S_1.$$

Suppose that we have an extension

$$h_i\colon S'_1 - \text{Int } S'_i \leftrightarrow \text{Pr } S_1 - \text{Int Pr } S_i,$$

such that

$$h_i(J'_{ij}) = \text{Pr } J_{ij} \qquad (1 \leqslant j \leqslant m_i).$$

We then extend h_i to all arcs

$$b'_{ijk} = d'_{ijk} \cap d'_{i,j+1,k} \subset A'''_{i,j+1},$$

so that these are mapped onto disjoint broken lines b_{ijk} in Pr A''_{ij}, intersecting Bd S_{i+1} in points p_{ijk}. For $k = 1, 2$, let e_{ijk} be disjoint broken lines in the annulus Bd Pr $S_{ij} \cap$ Bd Pr S_{i+1}, joining p_{ijk} to $p_{i,j+1,k}$, and intersecting the boundary of the annulus only in their end-points. We define an extension h_{i+1} of h_i such that

$$h_{i+1}(b'_{ijk}) = b_{ijk},\ h_{i+1}(d'_{ijk} \cap \text{Bd } S'_{i+1}) = e_{ijk}.$$

We choose the broken lines e_{ijk} in such a way that (1) $h_{i+1}(\text{Bd } d'_{ijk})$ is latitudinal in Pr S_{ij} and (2) each of the polygons $\bigcup_j e_{ijk}$ ($k = 1, 2$) intersects each set Pr $J_{i+1,r}$ in exactly one point. For $k = 1, 2$ let d_{ijk} be a polyhedral 2-cell in Pr S_{ij} such that

$$\text{Bd } d_{ijk} = h_{i+1}(\text{Bd } d'_{ijk});$$

we choose these 2-cells so that they are disjoint.

Now any homeomorphism between the boundaries of two cells can be extended so as to give a homeomorphism between the cells. Thus we can extend h_{i+1} so that $h_{i+1}(A'''_{ij}) = A''_{ij}$ for each j. The arcs $d'_{ijk} \cap \text{Bd } S'_{i+1}$ decompose each set $\text{Bd } S'_{i+1} \cap \text{Bd } S'_{ij}$ into two 2-cells, each of which intersects each set $J'_{i+1,r}$ in an arc. We extend h_{i+1} to each of these 2-cells, in such a way that

$$h_{i+1}(\text{Bd } S'_{i+1} \cap \text{Bd } S'_{ij}) = \text{Bd Pr } S_{i+1} \cap \text{Bd Pr } S_{ij};$$

and we choose these extensions in such a way that $h_{i+1}(J'_{i+1,r}) = \text{Pr } J_{i+1,r}$ for each r. Now $d'_{i,j,1} \cup d'_{i,j,2}$ decomposes S'_{ij} into two 3-cells, and h_{i+1} has already mapped the boundary of each of these onto a polyhedral 2-sphere in $\text{Bd Pr } S_{ij} \cup d_{i,j,1} \cup d_{i,j,2}$. Thus h_{i+1} can be extended to give

$$\begin{aligned} h_{i+1}(S'_{ij}) &= \text{Pr } S_{ij}, \\ h_{i+1}(S'_1 - \text{Int } S'_{i+1}) &= \text{Pr } S_1 - \text{Int Pr } S_{i+1}, \\ h_{i+1}(J'_{i+1,r}) &= \text{Pr } J_{i+1,r} \end{aligned}$$

for each r. Inductively, we get an ascending sequence $h_1, h_2, \ldots$ of homeomorphisms whose union is a homeomorphism

$$h: S'_1 - F' \leftrightarrow \text{Pr } S_1 - \text{Pr } F.$$

By continuity, we can extend h to F', so that $h(S'_1) = \text{Pr } S_1$; the verification depends on Theorem 11.4.

Evidently the above proof proves more than the theorem, namely:

THEOREM 11.6. $\text{Pr } S_1$ *has a triangulation* $K(\text{Pr } S_1)$ *such that* (1) $\text{Pr } F$ *forms a subcomplex of* $K(\text{Pr } S_1)$ *and* (2) *the identity mapping* $\text{Bd Pr } S_1 \leftrightarrow \text{Bd Pr } S_1$ *is* PL *relative to* $K(\text{Pr } S_1)$ *and* $K(U)$.

PROOF. Let $K(\text{Pr } S_1) = \{\sigma | \sigma = h(\tau), \tau \in K(S'_1)\}$.

THEOREM 11.7. Ω *has a triangulation* $K(\Omega)$ *in which* $\text{Pr } F$ *forms a subcomplex.*

PROOF. We have triangulations $K_1 = K(\text{Pr } S_1)$ and

$$K_2 = K(\Omega - \text{Int Pr } S_1) = \{\sigma | \sigma \in K(U), \sigma \subset \Omega - \text{Int Pr } S_1\};$$

and the relation between these in PL in $\text{Bd Pr } S_1$. Thus the two induced triangulations of $\text{Bd Pr } S_1$ have a common subdivision; and this subdivision can be extended to give subdivisions K'_1, K'_2 of K_1 and K_2. Let $K(\Omega) = K'_1 \cup K'_2$.

THEOREM 11.8. Ω *is a* 3-*manifold, and* $\text{Pr } F$ *is tame in* Ω.

Proof by Theorems 11.5 and 11.7.

12. Proof of Theorem 1.4. Let M, $K(M)$, and F be as in Theorem 1.4, and let D be a tame 2-cell such that $\operatorname{Bd} D = F$. Since D is tame, Int D is also tame. By Theorem 3.4 there is a homeomorphism $h\colon M \leftrightarrow M$ such that $h(\operatorname{Int} D)$ is a polyhedron (necessarily infinite) relative to $K(M)$ and $h|F$ is the identity. Let A be an annulus in $h(D)$, such that F is a component of Bd A and the other component of Bd A is also a polygon. Lemma (2.1) of [$\mathbf{M_8}$] asserts that for every neighborhood U of F there is a homeomorphism h': $M \leftrightarrow M$, such that (1) $h'(A)$ is a polyhedron and (2) h' is the identity at each point of F and at each point of $M - U$. (In fact, this lemma was stated for the case $M = \mathbf{R}^3$, but the corresponding result for arbitrary triangulated 3-manifolds follows by the same methods. Since D has a 3-cell neighborhood in M, the general result is also derivable from the special result, with the aid of the Hauptvermutung.)

It follows that F is the boundary of a 2-cell which is polyhedral relative to $K(M)$, which was to be proved.

A final remark. In §1 it may have seemed more natural to say that a tame 1-sphere is *unknotted* if it bounds a 2-cell (tame or not). But in fact it makes no difference. Given $J = \operatorname{Bd} D$, where J is tame and D is a 2-cell, let h be a homeomorphism $M \leftrightarrow M$ such that $h(J)$ is a polygon. We can then show that $h(J)$ is the boundary of a polyhedral 2-cell; the proof is a minor variation on the familiar process by which the Dehn Lemma is deduced from the Loop Theorem. (See, for example, [**MGT**, pp. 197–199].) Therefore J is the boundary of a tame 2-cell, and J is unknotted in the sense defined in §1.

BIBLIOGRAPHY

[$\mathbf{B_1}$] R. H. Bing and R. J. Bean (Ed.), *Topology seminar, Wisconsin*, 1965, Ann. of Math. Studies, No. 60, Princeton Univ. Press, Princeton, N. J., 1960, p. 82.

[$\mathbf{B_2}$] R. H. Bing, *An alternative proof that 3-manifolds can be triangulated*, Ann. of Math. (2) **69** (1959), 37–65.

[$\mathbf{B_3}$] ———, *Inequivalent families of periodic homeomorphisms of* E^3, Ann. of Math. (2) **80** (1964), 78–93.

[$\mathbf{B_4}$] Armand Borel, *Seminar on transformation groups*, Ann. of Math. Studies, No. 46, Princeton Univ. Press, Princeton, N. J., 1960.

[$\mathbf{B_5}$] Glen E. Bredon, *Orientation in generalized manifolds and applications to the theory of transformation groups*, Michigan Math. J. 7 (1960), 35–64.

[E] C. H. Edwards, *Concentricity in 3-manifolds*, Trans. Amer. Math. Soc. **113** (1964), 406–423.

[FA] Ralph H. Fox and Emil Artin, *Some wild cells and spheres in three-dimensional space*, Ann. of Math. (2) **49** (1948), 979–990.

[M] Edwin E. Moise, *Periodic homeomorphisms of the 3-sphere*, Illinois J. Math. **6** (1962), 206–225.

[$\mathbf{M_4}$] ———, *Affine structures in 3-manifolds*. IV. *Piecewise linear approximations of homeomorphisms*, Ann. of Math. (2) **55** (1952), 215–222.

[$\mathbf{M_5}$] ———, V. *The triangulation theorem and Hauptvermutung*, Ann. of Math. (2) **56** (1952), 96–114.

[M_8] ______, VIII. *Invariance of the knot-types; local tame imbedding*, Ann. of Math. (2) **59** (1954), 159–170.

[MGT] ______, *Geometric topology in dimensions* 2 *and* 3, Springer-Verlag, New York, 1977.

[P] C. D. Papakyriakopoulos, *On solid tori*, Proc. London Math. Soc. (3) **7** (1957), 248–260.

[S_1] P. A. Smith, *Transformations of finite period*. II, Ann. of Math. (2) **40** (1939), 690–711.

[S_2] ______, *Periodic transformations of* 3-*manifolds*, Illinois J. Math. **9** (1965), 343–348.

[St] John Stallings, *On the loop theorem*, Ann. of Math. (2) **72** (1960), 12–19.

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